STUDENT ARTICLE

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Dunking Donuts: Culinary Calculations of the Euler Characteristic

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Abstract

Motivated by a remarkable 18th-century result about polyhedra known as Euler's formula, we will develop the notion of the Euler characteristic χ in the more modern context of CW complexes. The fact that χ is a homotopy invariant gives an easy (perhaps trivializing) proof of Euler's formula. We then develop two non-elementary methods of computing χ in specific cases: Morse theory and the Poincaré-Hopf Index Theorem. Both will be used to compute the Euler characteristic of closed orientable surfaces, using culinary analogies. In an appendix, the former will also be used to compute the Euler characteristic of real projective space.

Most of this paper requires only an understanding of multivariable calculus and basic point-set topology. While the reader would be aided by a modest background in differential and algebraic topology at a few points, the degree of formality does not require this.‡

1.1 The Euler Characteristic and CW Complexes

The **Euler characteristic** $\chi(P)$ of a polyhedron P is defined to be the number F of its faces, minus the number E of its edges, plus the number V of its vertices:

$$\chi(P) = F - E + V.$$

We consider any n-sided polygon to be "filled in," so it has one face. Then we immediately have:

$$\chi(\text{any } n\text{-gon}) = 1 - n + n = 1.$$

We have easily seen that the Euler characteristic of a polygon is independent of the number and arrangement of these sides; less obviously, any convex polyhedron satisfies

$$\chi(\text{any convex polygon}) = 2.$$

This fact, known as **Euler's formula**, was known to Leonhard Euler (1707-1783), the namesake of χ. From Euler's formula, it is not hard to prove the classification of Platonic solids. (The original

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‡Diagrams for this article were created in METAPOST by Graphic Artist Zachary Abel '10, based on drawings submitted by the author.

classification argument, which proceeds by adding up angles at a vertex, appears in Book XIII of Euclid's *Elements*.)

There is a more modern definition of χ which generalizes it to a homotopy invariant of CW complexes. Once we see what a CW complex is, all this means is that stretching, bending, folding, and compressing our space will not change its Euler characteristic; we may not, however, cut or glue.

We will define the notion of a **CW complex** inductively. A zero-dimensional CW complex is just a set of points, also called the 0**-skeleton**. The data of a one-dimensional CW complex X is a 0-skeleton X_0, a set of closed 1-discs (closed intervals) $\{I_\alpha\}_{\alpha\in A}$, and a set of corresponding maps

$$\{\phi_\alpha : \partial I_\alpha \to X_0\}_{\alpha\in A}$$

taking the boundary of each closed 1-disc to the 0-skeleton. The complex X (or its 1-skeleton X_1) is then the quotient space

$$X = \left(X_0 \amalg \coprod_{\alpha\in A} I_\alpha \right) / \{\phi_\alpha\}_{\alpha\in A}.$$

(The symbol $\coprod$ just means a union of disjoint topological spaces, where the open sets are unions of open sets taken from either space.) When we quotient by a family of maps, we are quotienting by the equivalence relation which identifies each point of each ∂I_α with its image under the corresponding map ϕ_α. Geometrically, we are just attaching each closed 1-disc I_α to X_0 by gluing its endpoints to their images under ϕ_α. Inductively, an n-dimensional CW complex is given by an $(n-1)$-skeleton X_{n-1}, a set of $\{D_\beta\}_{\beta\in B}$ of closed n-discs,[1] and attaching maps $\{\psi_\beta : \partial D_\beta \to X_{n-1}\}_{\beta\in B}$. The complex is then the quotient space

$$X = \left(X_{n-1} \amalg \coprod_{\beta\in B} D_\beta \right) / \{\phi_\beta\}_{\beta\in B}.$$

Further details can be found in Chapter 0 of [Ha].

An example which will be useful in just a moment: the n-sphere $S^n = \{v \in \mathbb{R}^{n+1} : |v| = 1\}$ is homeomorphic to the CW complex given by:

one 0-cell, the point p

one n-cell D with attaching map $\phi(x) = p$ for all $x \in \partial D$.

In other words, we start with the closed n-disc D, and glue the entire bounding $(n-1)$-sphere to a point.

Now say we have an n-dimensional CW complex Y whose k-cells are given by the set C_k. Write $\mathrm{Card}(C_k)$ for the cardinality for C_k, that is, the number of k-cells. Furthermore, say that each C_k is a finite set. Then we define the Euler characteristic of Y to be

$$\chi(Y) = \sum_{k=0}^{n} (-1)^k \,\mathrm{Card}(C_k).$$

This generalizes our earlier definition, since vertices, edges, and faces can be taken to be the 0-, 1-, and 2-cells of a two-dimensional CW complex. It turns out (see section 2.2 of [Ha]) that χ is a homotopy invariant in the sense mentioned earlier.[2] In particular, homeomorphic CW complexes have the same χ,

[1] By n-disc, we simply mean a space homeomorphic to the unit ball in $\mathbb{R}^n$, that is, $\{v \in \mathbb{R}^n : |v| < 1\}$. When we add the adjective **closed**, we simply mean the closure in $\mathbb{R}^n$ of such a set.

[2] For those familiar with cellular homology, the proof is not hard. One can show purely algebraically that given a chain complex $C_0 \to C_1 \to C_2 \to \cdots$ of finitely generated abelian groups, $\sum(-1)^k \,\mathrm{rk}(C_k) = \sum(-1)^k \,\mathrm{rk}(H_k)$, where H_k is the k-th homology group of the complex. In the case of the cellular complex, C_k is simply a freely generated $\mathbb{Z}$-module with rank equal to the number of k-cells, so $\chi(X) = \sum(-1)^k \,\mathrm{rk}(C_k(X)) = \sum(-1)^k \,\mathrm{rk}(H_k(X))$. And since the Betti numbers $b_k = \mathrm{rk}(H_k(X))$ are homotopy invariants, so is the Euler characteristic $\chi(X)$.

since every homeomorphism is certainly a homotopy equivalence. Viewed conversely, we can compute χ of a given space by choosing a CW complex on it, and our computation will not depend on our choice of CW structure. (This is tautologous, since by "choosing a CW structure" on a space we merely mean finding a CW complex homeomorphic to our space.)

As a corollary to all of this, we have an immediate proof of Euler's formula, that all convex polyhedra "miraculously" have Euler characteristic equal to 2. Indeed, any convex polyhedron can be "smoothed out" by a homotopy equivalence (in fact a homeomorphism) into a 2-sphere. Then as explained above, the 2-sphere has one 0-cell and one 2-cell, and thus has Euler characteristic

$$\chi(S^2) = 1 - 0 + 1 = 2.$$

Similarly and more generally, we have

$$\chi(S^n) = \begin{cases} 0 & n \text{ is odd} \\ 2 & n \text{ is even.} \end{cases}$$

1.2 A Little Morse Theory

In a landmark 1934 paper [Mo], Marston Morse (1892-1977) initiated the theory which came to bear his name. The basic idea of Morse theory is to study a smooth manifold by a certain class of smooth functions on it, called Morse functions. It turns out that the typical smooth function is a Morse function.

Let M be a smooth (C^∞) manifold, and let $f : M \to \mathbb{R}$ be a smooth function on M. Recall that a **critical point** of f is a point p such that df_p is a degenerate linear map. In this case, this is equivalent to saying that in a local coordinate system $\{x_1, \ldots, x_n\}$ around p, all the first partial derivatives vanish:

$$p \text{ is a critical point of } f \quad \Leftrightarrow \quad \frac{\partial f}{\partial x_1}(p) = \ldots = \frac{\partial f}{\partial x_n}(p) = 0.$$

In single-variable calculus, we measure the behavior of a function at a critical point by looking at the sign of the second derivative, if non-vanishing. If the second derivative vanishes, we need to consider higher derivatives (think of $f_1(x) = x^3$ and $f_2(x) = x^4$ at $x = 0$). Analogously, we want to consider **non-degenerate** critical points, which are defined to be critical points where the matrix of second partial derivatives determines a non-degenerate bilinear form:

$$\text{the critical point } p \text{ of } f \text{ is non-degenerate} \quad \Leftrightarrow \quad \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right) \neq 0,$$

where i and j are the row and column indices. Then the class of functions which we can easily work with are those whose critical points are all non-degenerate; we call these **Morse functions**. The obvious generalization of looking at the sign of the single-variable first derivative is to look at the signs of the eigenvalues of df_p. However, this would force us to worry about existence of real eigenvalues, and this may not even be stable under change of coordinates. Instead we appeal to a famous and convenient result which guarantees a "nice" set of coordinates.

Lemma 1. *(The Morse Lemma.) Let p be a non-degenerate critical point of the smooth function $f : M \to \mathbb{R}$. Then there exists a neighborhood U of p and a coordinate system $\{y_1, \ldots, y_n\}$ on U centered at y such that on U,*

$$f(y) = f(p) \pm y_1^2 \pm y_2^2 \pm \ldots \pm y_n^2.$$

Furthermore, any such coordinate system will give the same numbers of positive and negative terms in the above.

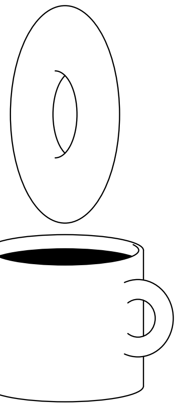

Figure 1.1: Dunking a donut (torus) into coffee

Since our focus is on different tools for computing the Euler characteristic and not on a rigorous development of Morse theory, we refer the reader to section 2 of [Mi] for a proof. We call the number of negative terms $\mathrm{MInd}(f;p)$, the **Morse index** of f at p; intuitively, the Morse index measures the number of independent directions in which f decreases.

For any real number a, let

$$M^a = f^{-1}((-\infty, a]).$$

The intuitive picture is a follows. Say we are dunking a donut into a cup of coffee, as in Figure 1; the manifold in question is the torus T which is the surface of this donut. Define the function $h : T \to \mathbb{R}$ by

$$\begin{aligned} h(p) &= \text{the height of the submerged part of } T \text{ when } p \text{ first touches the coffee} \\ &= \text{the vertical distance from the bottom of the donut to } p. \end{aligned}$$

We will call h, and its later generalizations, the "dunking function." It is not hard to check that h is a Morse function. Figure 2 shows T^a for various values of a. The set of critical points of h is $\{p_0, p_1, p_2, p_3\}$, as pictured. Their indices are:

$$\begin{aligned} \mathrm{MInd}(h;p_0) &= 0 \\ \mathrm{MInd}(h;p_1) &= 1 \\ \mathrm{MInd}(h;p_2) &= 1 \\ \mathrm{MInd}(h;p_3) &= 2. \end{aligned}$$

This is not hard to see: p_0 is a local (in fact, global) minimum, so any direction is a direction of increase, so it has index 0. p_1 decreases if you walk down towards p_0, and increases if you want up the inside of the hole towards p_2, so it has index 1. And so forth.

The first major application of the Morse index, and the one we care about for our purposes, is that it allows you to construct a CW complex homotopy equivalent to M.

Theorem 2. *Let p be a critical point of the Morse function $f : M \to \mathbb{R}$, and set $a = f(p)$. Suppose $f^{-1}([a-\epsilon, a+\epsilon])$ for some $\epsilon > 0$ is compact and contains no critical points other than p. Then $M^{a+\epsilon}$ has the homotopy type of $M^{a-\epsilon}$, with a cell of dimension $\mathrm{MInd}(f;p)$ adjoined.*

(For a proof, see section 3 of [Mi].) So a Morse function gives us a CW structure on M, up to homotopy equivalence. And since χ is a homotopy invariant, this is as good as we need. Combined with

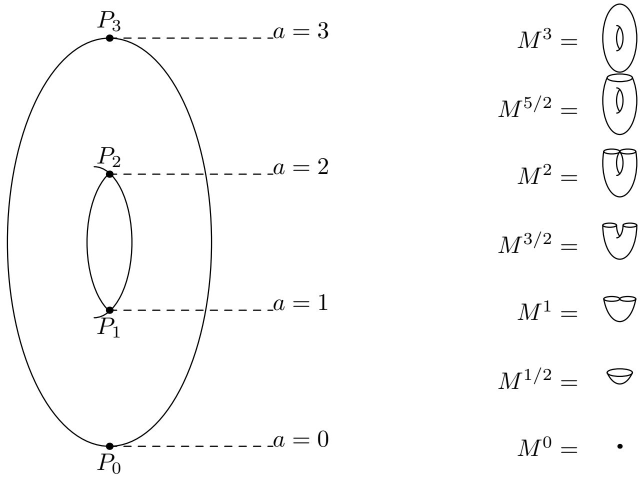

Figure 1.2: The torus, at and between the critical points of its dunking map

the fact that every smooth manifold admits Morse functions (see section 6 of [Mi]), we immediately obtain:

Corollary 3. *Every smooth manifold is homotopy equivalent to a CW complex.*

This implies that the Euler characteristic is defined for all smooth manifolds. If we set

$$A_k(f) = \text{the number of critical points of } f \text{ with Morse index } k$$

and apply Theorem 2, we have

$$\chi(M) = \sum_{k=0}^{n} (-1)^k A_k.$$

Define the surface Σ_g of genus g to be the surface of a g-holed donut; for example, $\Sigma_0 = S^2$ (a "donut hole") and $\Sigma_1 = T$. Consider the "dunking function" h above, but now more generally on any Σ_g; see Figure 3. h always has exactly one maximum and one minimum, and two saddle points (points of Morse index 1) for each hole; we have

$$\begin{aligned} A_0(h) &= 1 \\ A_1(h) &= 2g \\ A_2(h) &= 1 \\ \chi(\Sigma_g) &= 1 - 2g + 1 = 2 - 2g. \end{aligned}$$

1.3 Vector Fields and the Poincaré-Hopf Index Theorem

We now turn to smooth (tangent) vector fields on M. We will think of M as embedded in some $\mathbb{R}^N$ and the vector fields as tangent to $M \subset \mathbb{R}^N$ (if you are aware of the terminology, you may think more abstractly of the vector fields as sections of the tangent bundle TM). For this section only we restrict our attention to the case where M is two-dimensional, but we will indicate the correct generalization to higher dimensions.

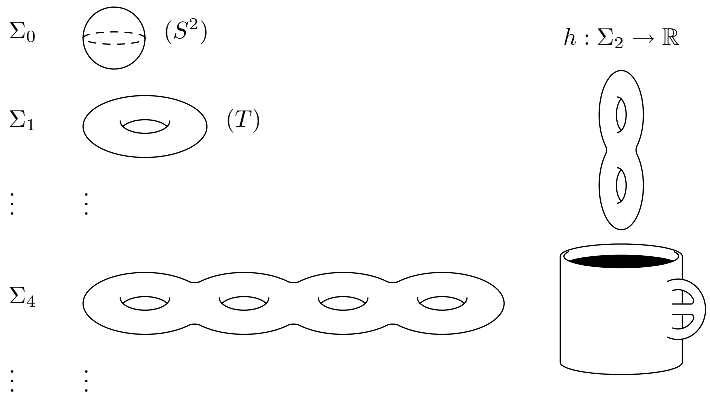

Figure 1.3: Surfaces of higher genus; dunking a two-holed donut into coffee

Let $M \subset \mathbb{R}^N$ be a smooth manifold embedded in Euclidean space. The vector space of vectors tangent to M at a point p, called the **tangent space** T_pM to M at p, is of the same dimension as M. Let $\{x_1, \dots, x_n\}$ be a smooth coordinate system for M centered at p; that is, p is the point for which $x_j = 0$ for all j. We write $(a_1, \dots, a_n)$ for the point with coordinate $a_j = x_j$. Then the tangent space can be written as

$$T_pM = \text{span}\{v_1, \dots, v_n\},$$

where

$$v_j = \frac{d}{dt}|_{t=0}(0, \dots, 0, t, 0, \dots, 0)$$

(the only non-zero entry is the j-th). The corresponding picture is that if we were to trace out a curve given by increasing only coordinate x_j, the vector $v_j \in T_pM$ would be the velocity vector of this curve as it passed p. If you are not comfortable or familiar with the language of tangent spaces, you may just picture these vectors as the tangent plane to a surface $M \subset \mathbb{R}^3$. We define a **vector field** on M to be a choice of vector $v(p) \in T_pM$ for each $p \in M$.

Let $v : M \to \mathbb{R}^N$ be a smooth vector field, and let p be an isolated zero of v. Let $y = (y_1, y_2)$ be a local set of coordinates centered at p, and choose a small circle S_ϵ of radius $\epsilon > 0$ centered at p in these coordinates. Then the map

$$\rho_v : S_\epsilon \to S^1$$
$$\rho_v(y) = \frac{v(y)}{|v(y)|}$$

can be defined, and we define the **local index** of v at p to be

$$\text{Ind}(v; p) = w(\rho_v).$$

Here, $w(\rho_v)$ is the winding number of ρ_v around S^1 (the net number of times ρ_v wraps around S^1 when we go around S_ϵ once, with counterclockwise being the positive sense).[3]

To see what local indices look like, consider Figure 4. If we walk around the small dotted-line circle centered at the zero of the vector field, we can see the local index by counting how many counterclockwise revolutions the arrows make. In example (a), the image $\rho_v(x)$ starts pointing to the right,

[3]For the topologically advanced: More generally, for $\dim(M) = n \geq 2$, S_ϵ is an $(n-1)$-sphere, and instead of $w(\rho_v)$, we use the topological degree of the map $\rho_v : S_\epsilon \to S^{n-1}$. One can prove that for ϵ small enough, the local index is well-defined. For more details, see chapter 3 of [Gu].

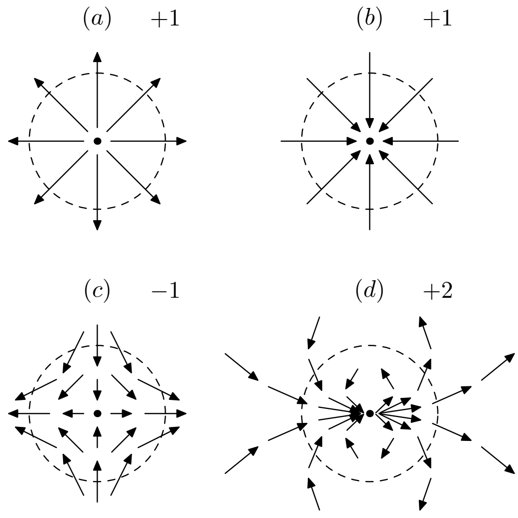

Figure 1.4: Local indices at zeroes of a vector field

then points upwards, then left, then down, and then right again; ρ_v has traversed S^1 once in the counterclockwise direction, so the local index is $+1$. So a "source" has index $+1$. Looking at (b), we see that a "sink" also has index $+1$: starting to the right of the zero, ρ_v starts pointing left, then down, then right, then up, and finally left again. It takes something like the situation in (c) to get a negative local index. Doing the same sort of walk around, ρ_v starts pointing right, then down, then left, then up, and finally right again; we have traversed S^1 once in the clockwise direction. Example (d) shows a local index of $+2$. If v has finitely many zeroes, then we define the **global index** (or simply the **index**) of v to be the global sum of its local indices:

$$\text{Ind}(v) = \sum_{v(x)=0} \text{Ind}(v; x).$$

The remarkable namesake of this section is the following:

Theorem 4. *(The Poincaré-Hopf Index Theorem.) Let v be a smooth vector field on M with finitely many zeroes. Then the global index of v equals the Euler characteristic of M:*

$$\text{Ind}(v) = \chi(M).$$

The two-dimensional case was proved by Jules Henri Poincaré (1854-1912) in 1885; Heinz Hopf (1894-1971) proved the general case in 1927. In particular, the full Poincaré-Hopf Index Theorem predates Morse theory. A proof using Morse theory, however, is popular; see chapter 12 of [Ma]. For a proof using the Lefschetz fixed point theorem, see chapter 3 of [Gu]. The immediate corollary of this theorem is that the global index is the same, regardless of which vector field you choose; this is analogous to the fact that the alternating sum $\sum(-1)^k A_k$ did not depend on the choice of Morse function.

We now use the Poincaré-Hopf Index Theorem to compute again the Euler characteristic of the surface Σ_g. The vector field we will choose is again culinary: the "hot fudge vector field" v_{hf} depicted in Figure 5. Simply stand Σ_g on end as shown, and pour hot fudge over the surface. In an ideal steady state situation, all the fudge enters at one point on top, and all the fudge drips off at one point on the bottom. Then define the value of v_{hf} at a point to be the instantaneous velocity vector of the hot fudge flow at that point. We have a source at the top and a sink at the bottom (neglecting the inflow and outflow, which are not tangent to the surface), and saddle points (points which look like Figure 4c) at the top and bottom of each hole (you should try to picture this yourself). We saw earlier that sources and sinks have index $+1$ and saddles have index -1, so we conclude

$$\chi(\Sigma_g) = \mathrm{Ind}(v_{hf}) = 1 + (2g)(-1) + 1 = 2 - 2g.$$

If you compare how the computations went here and in the section on Morse theory, in both cases each hole contributed two "negative units" (odd dimensional CW cells or negative index zeroes), and the two ends each contributed one "positive unit." Since the computations are similar in nature, it makes sense that one is able to prove the Poincaré-Hopf Index Theorem using Morse theory.

We conclude this section with a corollary, which contains a famous and amusingly named result as a special case.

Corollary 5. *A smooth manifold M with $\chi(M) \neq 0$ does not admit a smooth, nowhere vanishing vector field.*

Proof. Let v be a smooth vector field on M. Then by the Poincaré-Hopf Index Theorem, $\mathrm{Ind}(v) = \chi(M) \neq 0$. If v were nowhere vanishing, the sum defining $\mathrm{Ind}(v)$ would be empty, forcing $\mathrm{Ind}(v) = 0$; this is impossible. □

Corollary 6. *The surface Σ_g of genus g admits a nowhere vanishing smooth vector field if and only if $g = 1$, that is, if and only if Σ_g is the torus.*

Proof. The "only if" direction is immediate from the previous corollary and our earlier computation, $\chi(\Sigma_g) = 2 - 2g$. Conversely, we can construct a nowhere vanishing vector field on the torus by the process depicted in Figure 6: first take a nowhere vanishing vector field on S^1, and then revolve the entire construction about an axis away from it. □

The special case $g = 0$, that is $\Sigma_0 = S^2$, is known as the "Hairy Ball Theorem." Intuitively, it states that there is always at least one point on the surface of Earth with no wind blowing. Equivalently, if the Earth had hair, it would necessarily have a bald spot.

1.4 An Example: Real Projective Space

Define **real projective space**[4] **of dimension** n to be the quotient space

$$\mathbb{RP}^n = \mathbb{R}^{n+1} - \{0\}/\sim$$
$$v \sim w \quad \Leftrightarrow \quad v = \lambda w \text{ for some } \lambda \in \mathbb{R} - \{0\}.$$

Since the equivalence class of a non-zero vector v is the one-dimensional subspace of $\mathbb{R}^{n+1}$ spanned by v (minus the point 0) and every one-dimensional subspace contains a non-zero vector, $\mathbb{RP}^n$ is just the set of one-dimensional subspaces of $\mathbb{R}^{n+1}$, topologized.

Note that a particular one-dimensional subspace $U \subset \mathbb{R}^{n+1}$ intersects the unit sphere $S^n \subset \mathbb{R}^{n+1}$ in exactly two points, namely the two vectors $v, -v$ of length 1 in U. Thus any even function[5] on S^n determines a function on $\mathbb{RP}^n$; it is easy to check that if such a function is smooth on S^n, it is smooth on $\mathbb{RP}^n$ as well. Let $\{a_0, a_1, \ldots, a_n\}$ be an ordered set of distinct, non-zero real numbers; for simplicity, assume they are in ascending order. Define the function

$$f : S^n \to \mathbb{R}$$
$$f(x) = a_0x_0^2 + a_1x_1^2 + \ldots + a_nx_n^2;$$

[4]We borrow greatly from chapter 12 of [Ma] for the first half of this section.

[5]Recall that a function $f : V \to X$ on a vector space V is said to be **even** if $f(v) = f(-v)$ for all $v \in V$.

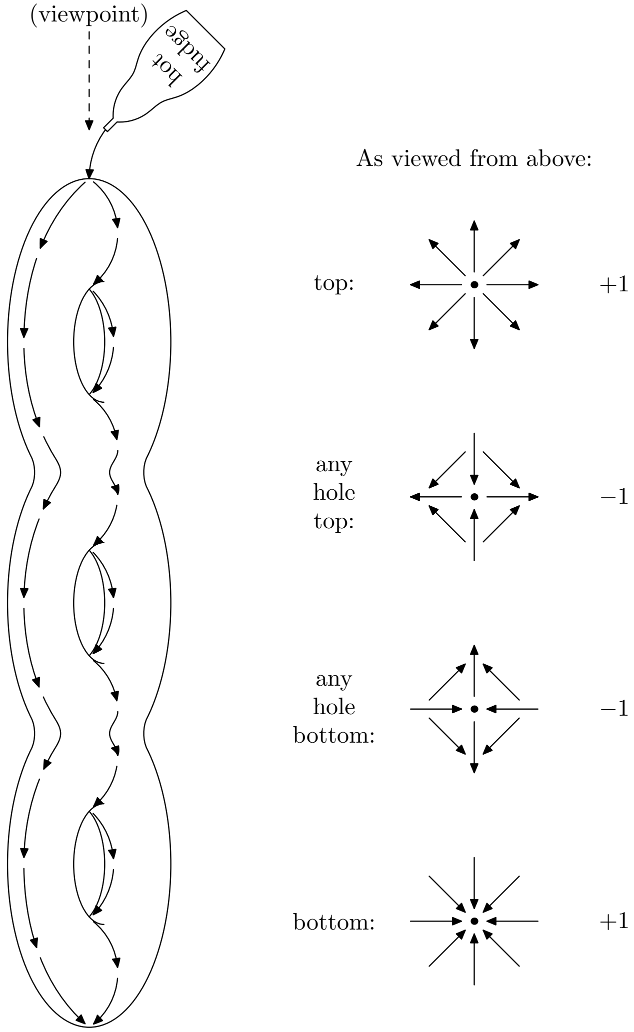

Figure 1.5: A vector field constructed by coating a three-holed donut in hot fudge

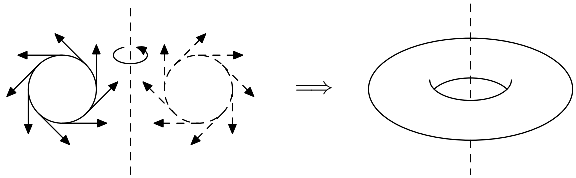

Figure 1.6: Constructing a non-vanishing vector field on the torus

using the standard coordinates $\{x_0, \ldots, x_n\}$ on $\mathbb{R}^{n+1}$. Since f is even, it determines a function on $\mathbb{RP}^n$, which by abuse of notation, we also call f.

We will determine and classify the critical points of f, conclude that it is a Morse function, and use this to build a CW structure on $\mathbb{RP}^n$. Afterwards, we will re-construct this CW structure in a more elementary fashion. As a corollary of either approach, we will compute $\chi(\mathbb{RP}^n)$.

Since the properties of f at a point are local in nature, we can continue working with the explicit embedding $S^n \subset \mathbb{R}^{n+1}$. At the point $x = (x_0, \ldots, x_n)$, the tangent space is

$$T_x S^n = \{v = (v_0, \ldots, v_n) \subset \mathbb{R}^{n+1} : \sum x_i v_i = 0\},$$

and the first partial derivatives are given by

$$\frac{\partial f}{\partial x_i} = 2a_i x_i.$$

However, these are the partial derivatives with respect to the coordinates of the ambient space, $\mathbb{R}^{n+1}$. We do not need all of them to vanish; we merely need the gradient vector to be orthogonal to all vectors in the tangent space (for the more advanced: we need the differential to be the zero linear functional). In other words, we need to show that

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i} v_i = 0 \text{ for all } v = (v_0, \ldots, v_n) \in T_x S^n.$$

Since $|x| = 1$, it is impossible for the partial derivatives to all simultaneously vanish due to x being zero; instead, we use the relation $x \cdot v = 0$ for all $v \in T_x S^n$. The above equation holds, then, if and only if $x = (x_0, \ldots, x_n)$ and $(a_0 x_0, \ldots, a_n x_n)$ are parallel. But since the a_i are all distinct, this occurs if and only if $x = \pm e_i$, where e_i is the vector of all zeroes, except for a 1 in the i-th place. There are $2(n+1)$ such points on S^n, but only $n+1$ on $\mathbb{RP}^n$, since $e_i \sim -e_i$.

We now check that e_0 is a nondegenerate critical point and compute its Morse index. A local coordinate system $\{y_1, \ldots, y_n\}$ is defined by

$$(y_1, \ldots, y_n) \in \mathbb{R}^n \leftrightarrow \left(\pm\sqrt{1 - \sum y_i^2}, y_1, \ldots, y_n \right) \in S^n.$$

In terms of these coordinates, f looks like

$$f(y_1, \ldots, y_n) = a_0 \left(1 - \sum_{i=1}^{n} y_i^2 \right) + \sum_{i=1}^{n} a_i y_i^2 = a_0 + \sum_{i=1}^{n} (a_i - a_0) y_i^2.$$

The matrix of second partial derivatives is just

$$\begin{pmatrix} 2(a_1 - a_0) & & & \\ & 2(a_2 - a_0) & & \\ & & \ddots & \\ & & & 2(a_n - a_0) \end{pmatrix}.$$

Since the a_i are all distinct, the matrix is invertible, and $\pm e_0$ is a nondegenerate critical point. Also, the chosen coordinates are evidently of the form the Morse lemma guarantees, so we can read off the Morse index. Since the points were chosen to be in ascending order, each $a_i - a_0$ is positive and $\text{MInd}(f; \pm e_0) = 0$. The same analysis holds for each $\pm e_k$, with the exception that $(a_0 - a_k), (a_1 - a_k), \ldots, (a_{k-1} - a_k)$ will all be negative. Thus in the general case, we have

$$\text{MInd}(f; \pm e_k) = k.$$

Then the resulting CW structure on $\mathbb{RP}^n$ has one cell in each dimension from 0 through n inclusive, and

$$\chi(\mathbb{RP}^n) = \begin{cases} 1 & n \text{ is even} \\ 0 & n \text{ is odd.} \end{cases}$$

Finally, we give an elementary, geometric construction of this same CW structure. We begin by introducing **homogeneous coordinates** on $\mathbb{RP}^n$; while not coordinates in the usual sense, they are a convenient way of working explicitly in $\mathbb{RP}^n$. We will use $(n+1)$-tuple notation for the $\mathbb{R}^{n+1}$ our copy of $\mathbb{RP}^n$ is obtained from. The homogeneous coordinate for the point $p \in \mathbb{RP}^n$ is $[x_0, \ldots, x_n]$, where $x = (x_0, \ldots, x_n)$ is any non-zero vector in the one-dimensional subspace p of $\mathbb{R}^{n+1}$. In other words, in homogeneous coordinates,

$$[x_0, \ldots, x_n] = [y_0, \ldots, y_n] \quad \Leftrightarrow \quad x_i = \lambda y_i \text{ for all } i, \text{ and } \lambda \neq 0.$$

Also, a bracketed $(n+1)$-tuple $[x_0, \ldots, x_n]$ represents a point of $\mathbb{RP}^n$ if and only if not all its entries are zero.

Define an open subset $U_0 \subset \mathbb{RP}^n$ by

$$U_0 = \{[x_0, \ldots, x_n] \in \mathbb{RP}^n : x_0 \neq 0\}.$$

This is well-defined because nonzero scalar multiplication does not depend upon whether or not $x_0 = 0$, and it is open because it is the inverse image of $\mathbb{R} - \{0\}$ under the even, continuous map on S^n taking each point to the absolute value of its e_0 coordinate. The smooth map $\mathbb{R}^n \to U_0$ given by

$$(x_1, \ldots, x_n) \mapsto [1, x_1, \ldots, x_n]$$

has smooth two-sided inverse

$$[x_0, \ldots, x_n] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

This is well-defined because $x_0 \neq 0$, and because scaling all entries on the left does not affect the values on the right. Thus U_0 is diffeomorphic to $\mathbb{R}^n$; it is an n-cell. In order to determine what $\mathbb{RP}^n - U_0$ is, note that a bracketed $(n+1)$-tuple $[x_0, \ldots, x_n]$ is in $\mathbb{RP}^n - U_0$ if and only if $x_0 = 0$ but not all entries are zero; equivalently, a point of $\mathbb{RP}^n - U_0$ is just a choice of $x_1, \ldots, x_n$, not all zero. In other words, this complement is nothing other than a copy of $\mathbb{RP}^{n-1}$. We have found that

$$\mathbb{RP}^n = D^n \cup \mathbb{RP}^{n-1},$$

where we write D^k for an open k-dimensional cell. Noting that $\mathbb{RP}^0$ is just a point and inducting downwards,

$$\mathbb{RP}^n = D^0 \cup D^1 \cup \cdots \cup D^n.$$

This is the desired CW structure. Intuitively, $\mathbb{RP}^n$ contains an n-dimensional plane, and a copy of $\mathbb{RP}^{n-1}$ "at infinity"; this $\mathbb{RP}^{n-1}$ represents all possible directions in $\mathbb{R}^n$, up to identifying opposite directions. For instance, the projective plane contains the ordinary plane, as well as a circle's worth ($\mathbb{RP}^1 = S^1$) of infinities, each point on this circle being a direction in which you can go off to infinity from the plane.

1.5 Conclusion

To recap: as early as Euler, the curious observation had been made that the quantity $F - E + V$ corresponding to a convex polyhedron always equals 2. This so-called Euler characteristic was computed for other sorts of shapes, and results about it were proven, but it was not until the machinery of homotopy invariance became available that these results became "trivial" to prove. Indeed, any convex polyhedron can be "smoothed out" into a sphere, which has Euler characteristic 2.

In cases where we cannot immediately see what the Euler characteristic is by such a geometric trick, we can employ more sophisticated methods in our computations. The results of Morse and of Poincaré and Hopf that we have encountered tell us that given *almost any* vector field or smooth function on a manifold, we can compute the Euler characteristic of that manifold; viewed conversely, we can read these theorems as describing a topological constraint on any vector fields (with finitely many zeroes) or smooth (Morse) functions which may appear on a given manifold.

References

[Gu] Victor Guillemin and Alan Pollack: *Differential Topology*. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1974.

[Ha] Allen Hatcher: *Algebraic Topology*. Cambridge: Cambridge Univ. Press, 2002.

[Ma] Ib Madsen and Jørgen Tornehave: *From Calculus to Cohomology: De Rham cohomology and characteristic classes*. Cambridge: Cambridge Univ. Press, 1997.

[Mi] John Milnor: *Morse Theory*, Princeton, N.J.: Princeton Univ. Press, 1969 (Annals of Mathematics Studies **51**).

[Mo] Marston Morse: *The Calculus of Variations in the Large*. Providence, R.I.: American Math. Society, 1934. (Colloquium Publications **18**).