FEATURE

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Problems

The HCMR welcomes submissions of original problems in any field of mathematics, as well as solutions to previously proposed problems. Proposers should direct problems to Problems Editor Zachary Abel at hcmr-problems@hcs.harvard.edu or at the address on the inside front cover. A complete solution or a detailed sketch of the solution should be included, if known. Solutions to previous problems should also be directed to the Problems Editor at hcmr-solutions@hcs.harvard.edu or at the address on the inside front cover. Solutions should include the problem reference number, as well as the solver's name, contact information, and affiliated institution. Additional information, such as generalizations or relevant references, is also welcome. All correct solutions will be acknowledged in future issues, and the most outstanding solutions received will be published. To be considered for publication, solutions to the problems below should be postmarked no later than *November 1, 2007*. An asterisk beside a problem or part of a problem indicates that no solution is currently available.

S07 – 1. How many hyperplane cuts are necessary to divide a $3 \times 5 \times 7 \times 9 \times 11$ rectangular solid into $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$ distinct $1 \times 1 \times 1 \times 1 \times 1$ hypercubes, if previously separated pieces can be rearranged between cuts?

Proposed by Joel Lewis '07.

S07 – 2. Suppose $f : [0, 1] \to \mathbb{R}$ is an integrable function such that $y \cdot f(x) + x \cdot f(y) \le x^2 + y^2$. Show that $\int_0^1 f(x)\,dx \le \frac{\pi}{4}$. (One example of such a function is $f(x) = x$.)

Proposed by Scott Kominers '09.

S07 – 3. The incircle Ω_{ABC} of a triangle ABC is tangent to BC, CA, AB at P, Q, R respectively. Rays PQ and BA intersect at M, rays PR and CA intersect at N, and the incircle Ω_{MNP} of triangle MNP is tangent to MN and NP at X and Y respectively. Given that X, Y and B are collinear, prove:

(a) Circles Ω_{ABC} and Ω_{MNP} are congruent, and

(b) these circles intersect each other in $60°$ arcs.

Proposed by Zachary Abel '10.

S07 – 4. For a prime p, let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ denote the localization of the integral domain $\mathbb{Z}$ at the prime ideal (p); that is, the subring of $\mathbb{Q}$ consisting of the rational numbers with denominators prime to p. The canonical homomorphism $\mathbb{Z} \to \mathbb{F}_p$ induces a canonical homomorphism $\phi_p : \mathbb{Z}_{(p)} \to \mathbb{F}_p$, the reduction modulo p homomorphism with kernel the maximal ideal $p\mathbb{Z}_{(p)}$ of the local ring $\mathbb{Z}_{(p)}$. (For example, $\phi_5(1/2) = 3 \in \mathbb{F}_5$.)

Let V be the set of primes p for which $\{\frac{3^n-1}{2^n-1} \mid n \in \mathbb{N}\} \subset \mathbb{Z}_{(p)}$.

(a) Characterize the set V.

(b) Show that V and $P \setminus V$ are both infinite sets, where P is the set of primes. (In other words, show that V is neither finite nor cofinite in the set of primes.)

(c) Show that, for every $p \in V$, the map $\mathbb{N} \to \mathbb{F}_p$ given by $n \mapsto \phi_p\big((3^n - 1)/(2^n - 1)\big)$ is periodic. (For example, $5 \in V$, and the corresponding map $\mathbb{N} \to \mathbb{F}_5$ is $2, 1, 3, 2, 2, 1, 3, 2, 2, 1, 3, 2, \ldots$.)

Proposed by Vesselin Dimitrov '09.

S07 – 5. (a) Prove that, for distinct positive real numbers a and b, the following inequality holds:

$$\frac{a+b}{2} \geq \frac{a^{\frac{a}{a-b}} b^{\frac{b}{b-a}}}{e} \geq \frac{a-b}{\ln a - \ln b}.$$

(b*) Show that both inequalities are strict.

Proposed by Shrenik Shah '09.
