STUDENT ARTICLE

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Quivers

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Abstract

This project is based on the study of two kinds of representation theory: quiver representation theory and Lie algebra representation theory. By looking at some simple examples, we'll show how the two are connected. Indeed, we'll identify the isomophism classes of simple and indecomposable representations of a particular quiver with relation with the equivalence classes of simple and indecomposable representations of $\mathfrak{sl}_2(\mathbf{k})$. Throughout this paper, $\mathbf{k}$ will indicate an algebraically closed field of characteristic 0.[§]

3.1 Quivers

3.1.1 Definitions

A **quiver** is directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices (which is assumed to be finite) and Q_1 the set of arrows, with maps $h, t : Q_1 \to Q_0$ which assign to each arrow its head and tail, respectively. Every vertex $i \in Q_0$ has an associated edge e_i such that $h(e_i) = t(e_i) = i$.

A **path** is a sequence of arrows $p = a_1 a_2 \cdots a_n$ such that $h(a_{k+1}) = t(a_k)$ for $k = 1, \ldots, n-1$. The **head** of the path is $h(a_1)$, and the **tail** of the path is $t(a_n)$. Each e_i (defined above) is a trivial path which starts and ends at the vertex i.

An **oriented cycle** is a path p such that $h(p) = t(p)$, and $h(a_i) \neq t(a_j)$ for any other $i \neq j+1$.

It is easy to see the following property:

Proposition 1. *A quiver with an oriented cycle has an infinite set of paths.*

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Given a quiver Q, the **path algebra $\mathbf{k}Q$** is the $\mathbf{k}$-vector space generated by all paths in Q with multiplication rule:

$$p * q = \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

Given a point $i \in Q_0$, we have that e_i is the null path beginning and ending at that point, so $a * e_i = a$ whenever $t(a) = i$ and $e_i * b = b$ whenever $h(b) = i$. Note that the path algebra has a unit given by $\sum_{i \in Q_0} e_i$.

Example 3.1.1. The **Jordan quiver** J has one vertex $J_0 = \{1\}$ and one nontrivial arrow $J_1 = \{e_1, a\}$, such that $t(a) = h(a) = 1$.

Its path algebra has basis $\{e_1, a, a^2, \ldots\}$ and is thus infinite-dimensional.

Example 3.1.2. The **2-Kronecker** quiver K_2

$$\begin{tikzcd}
e_1 \bullet \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \bullet\, e_2
\end{tikzcd}$$

has finite-dimensional path algebra with basis $\{e_1, e_2, a, b\}$.

However, the **cyclic 2-Kronecker** quiver C_2 presented by

$$\begin{tikzcd}
e_1 \bullet \arrow[r, shift right, "b"'] & \bullet\, e_2 \arrow[l, shift right, "a"']
\end{tikzcd}$$

has an oriented cycle, and its path algebra is infinite-dimensional with basis

$$\{e_1, e_2, a, b, ba, ab, aba, bab, \ldots, (ba)^k, (ab)^k, a(ba)^k, b(ab)^k, \ldots\}.$$

Example 3.1.3. The **3-Kronecker** quiver K_3 presented by

$$\begin{tikzcd}
e_1 \bullet \arrow[r, shift left=2, "a" description] \arrow[r, "b" description] \arrow[r, shift right=2, "c" description] & \bullet\, e_2
\end{tikzcd}$$

has finite-dimensional path algebra with basis $\{e_1, e_2, a, b, c\}$.

3.1.2 Quivers with relations

We can impose further relations on the composition of arrows. This is equivalent to quotienting the path algebra by the apropriate ideal.

Example 3.1.4. For the Jordan quiver defined above, we can impose the relation $a^k = e_1$ for some $k \in \mathbb{N}$. The resulting path algebra has basis $\{e_1, a, a^2, \ldots, a^{k-1}\}$.

Example 3.1.5. For the quiver C_2 defined above, we can impose the relation $ab = e_2$, to obtain a quiver with path algebra basis given by $\{e_1, e_2, a, b, ba\}$.

3.2 Quiver Representations

3.2.1 Definitions

Given a quiver Q, a **quiver representation** of Q is a collection $\{V_i | i \in Q_0\}$ of finite dimensional $\mathbf{k}$-vector spaces together with a collection $\{\phi_a : V_{t(a)} \to V_{h(a)} | a \in Q_1\}$ of $\mathbf{k}$-linear maps such that $\phi_a \phi_b = \phi_{ab}$.

From now on, we will denote a representation by $\mathcal{R} = (\{V_i\}, \{\phi_a\})$.

Example 3.2.1. The representations $(\{V\}, \{\phi\})$ of the Jordan quiver are given by all $n \times n$ matrices A_ϕ, where $n = \dim V$.

Suppose $\mathcal{R} = (\{V_i\}, \{\phi_a\})$ and $\mathcal{R}' = (\{W_i\}, \{\psi_a\})$ are representations of Q. Then $\mathcal{R}'$ is a **subrepresentation** of $\mathcal{R}$ if

- for every $i \in Q_0$, W_i is a subspace of V_i and
- for every $a \in Q_1$, the restriction of $\phi_a : V_{t(a)} \to V_{h(a)}$ to $W_{t(a)}$ is equal to $\psi_a : W_{t(a)} \to W_{h(a)}$.

The **zero representation** of Q is given by $(\{V_i\}, \{\phi_a\})$ such that $V_i = 0$ for all $i \in Q_0$ and ϕ_a is the zero map for all $a \in Q_1$. A non-zero representation $\mathcal{R}$ is called **simple representation** if the only subrepresentations of $\mathcal{R}$ are the zero representation and $\mathcal{R}$ itself.

If $\mathcal{R} = (\{V_i\}, \{\phi_a\})$ and $\mathcal{S} = (\{W_i\}, \{\psi_a\})$ are representations of Q then we can define the **direct sum representation** $\mathcal{R} \oplus \mathcal{S} = (\{U_i\}, \{\rho_a\})$ by taking:

- $U_i = V_i \oplus W_i$ for every $i \in Q_0$, and
- $\rho_a : V_{t(a)} \oplus W_{t(a)} \to V_{h(a)} \oplus W_{h(a)}$, given by the matrix $\begin{pmatrix} \phi_a & 0 \\ 0 & \psi_a \end{pmatrix}$.

If $\mathcal{R}$ and $\mathcal{S}$ are two representations of Q, then a **representation morphism** $\Phi : \mathcal{R} \to \mathcal{S}$ is a collection of **k**-linear maps $\{\varphi_i : V_i \to W_i | i \in Q_0\}$ such that the diagram

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "\phi_a"] \arrow[d, "\varphi_{t(a)}"'] & V_{h(a)} \arrow[d, "\varphi_{h(a)}"'] \\
W_{t(a)} \arrow[r, "\psi_a"'] & W_{h(a)}
\end{tikzcd}$$

commutes for all $a \in Q_1$. If φ_i is invertible for every $i \in Q_0$, then the morphism Φ is called an **isomorphism** and $\mathcal{R}$ and $\mathcal{S}$ are called **isomorphic representations**.

A representation $\mathcal{R}$ of a quiver Q is called **decomposable** if $\mathcal{R} \simeq \mathcal{S} \oplus \mathcal{T}$ where $\mathcal{S}$ and $\mathcal{T}$ are nonzero subrepresentations of Q. A nonzero representation is called **indecomposable** if it cannot be written as such a direct sum. For any quiver Q, the simple representations of Q form a subclass of its indecomposable representations.

3.2.2 Isomorphism Classes of Representations

The study of quiver representations is significantly simplified if we consider isomorphism classes of quiver representations rather than representations themselves. To find a representative element of each isomorphism class, we apply representation isomorphisms to change the basis of the vector space at each vertex in order to simplify the matrices for the maps at each arrow. For representations over $\mathbb{C}$ with equidimensional vector spaces at each vertex, this process is the same as that of the Jordan normal form.

Example 3.2.2. For the Jordan quiver with one vertex and one arrow, every isomorphism class of representations has a representative element of the form $\mathcal{R} = (\{V_1\}, \{J_1\})$ where J_1 is a matrix in Jordan normal form and V_1 is a vector space with the associated basis.

This is a direct consequence of the theorem that every square matrix $M = P^{-1}JP$, where J is in Jordan form and P is an invertible matrix corresponding to the change of basis required to isolate the eigenspaces of the operator; see [Ha].

If we restrict ourselves to representations with invertible maps at each arrow, we may simultaneously describe the isomorphism classes of representations of quivers which differ from each other only in the orientation of their arrows. Note that the invertibility condition implies that the representation must have equidimensional vector spaces at all vertices. The isomorphism classes of these representations can often be described neatly, by analogy to the case of the Jordan quiver.

Example 3.2.3. Given any representation $\mathcal{R}$ of the cyclic 2-Kronecker quiver C_2 of the form

$$\mathcal{R} = (\{V_1, V_2\}, \{A, B\}),$$

where A and B are both invertible, we can find an isomorphic representation of the form:

$$\begin{tikzcd}
V_1 \bullet \arrow[r, shift right, "B"'] & \bullet V_2 \arrow[l, shift right, "A"'] \arrow[r, phantom, "\cong"] & V_1' \bullet \arrow[r, shift right, "J"'] & \bullet V_2' \arrow[l, shift right, "Id"']
\end{tikzcd}$$

with J in Jordan form. To find this isomorphic representation, let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for V_1 and V_2, respectively. Take P_0 to be the change-of-basis matrix taking $\mathcal{B}_2$ to $A\mathcal{B}_1$. This is possible because invertibility implies equidimensionality. Then the representation isomorphism $\Phi_0 = (Id, P_0)$ yields the isomorphic representation

$$\mathcal{R}' = (\{V_1', V_2'\}, \{Id, BA\}).$$

We can find an invertible matrix P_1 such that $BA = P_1^{-1} J P_1$, where J is a Jordan-form matrix. Applying the representation isomorphism $\Phi_1 = (P_1, P_1)$ yields the desired isomorphic representation.

Since we are considering only representations $(\{V_i\}, \{\phi_a\})$ with ϕ_a invertible for all $a \in Q_1$, quivers that differ only in the direction of their arrows (such as K_2 and C_2) have the same sets of representations. However, these quivers still have different representation theories (for example, different classes of simple representations), because the definition of a subrepresentation depends on the direction of arrows.

Example 3.2.4. The case of the 3-Kronecker quiver K_3 is more complicated than that of the 2-Kronecker, because we may not be able to simultaneously put the maps on the second and third arrows in Jordan normal form. However, in the case $\dim(V_1) = \dim(V_2) = 2$, we will always be able to conjugate bases and obtain an isomorphic representation of the form:

$$\begin{tikzcd}
V_1 \bullet \arrow[r, shift left=2, "\phi_a"] \arrow[r, "\phi_b" description] \arrow[r, shift right=2, "\phi_c"'] & \bullet V_2 \arrow[r, phantom, "\cong"] & V_1' \bullet \arrow[r, shift left=2, "Id"] \arrow[r, "J" description] \arrow[r, shift right=2, "\begin{pmatrix} i & 0 \\ j & k \end{pmatrix}"'] & \bullet V_2'
\end{tikzcd}$$

3.2.3 Simple Representations

Definition 2. An **i-th canonical representation** $\mathcal{R}_i$ for the quiver $Q = (Q_0, Q_1)$ is a representation of the form

$$\mathcal{R} = (\{V_j = \delta_{ij}\mathbf{k}\}, \{\phi_a = 0 \text{ for all } a \in Q_1\}).$$

where $\delta_{ij} = 1$ when $i = j$, $\delta_{ij} = 0$ when $i \neq j$.

Proposition 3. *Let Q be a quiver with no oriented cycles. A representation $\mathcal{R}$ of Q is simple if and only if it is canonical.*

Proof. $(\Leftarrow)$ A canonical representation $\mathcal{R}$ must be simple, because its only proper subrepresentation is the zero representation.

$(\Rightarrow)$ We will show that every non-canonical representation has a canonical subrepresentation.

Lemma 4. *If $Q = (Q_0, Q_1)$ is a quiver with no oriented cycles, then there is some vertex $i \in Q_0$ such that $i \neq t(a)$ for all arrows $a \in Q_1$. Such an arrow is called a* ***sink****.*

Proof. Suppose for every $v_i \in Q_0$, $v_i = t(a)$ for some $a \in Q_1$. Choose some $v_1 \in Q_0$ and form a path as follows: Choose a_n such that $t(a_n) = v_n$. Write $v_{n+1} = h(a_n)$, and repeat. As Q_0 is a finite set, eventually we will get $v_{n+1} = v_i$ for some $i \leq n$. Then $p = a_i \cdots a_n$ is an oriented cycle in Q. But by assumption, Q has no oriented cycles, so some vertex in Q must be a sink. □

So let Q be a quiver with no oriented cycle, let $x_1 \in Q_0$ be a vertex such that $t(a) \neq x_1$ for all $a \in Q_1$. Given an arbitrary representation $R = (V_i, \rho_a)$, if $V_{x_1} \neq \{0\}$, then write $x_n = x_1$ and proceed to the construction of $\mathcal{S}$ below.

If $V_{x_1} = \{0\}$, define $Q' = (Q'_0 = Q_0 \backslash \{x_1\}, Q'_1 = Q_1 \backslash \{a \in Q_1 | h(a) \neq x_1\})$. As Q contained no oriented cycles, and $Q'_0 \subset Q_0$, $Q'_1 \subset Q_1$, Q' contains no oriented cycle, so we may apply the lemma.

So we may let $x_2 \in Q'_0$ be a vertex such that $t(a) \neq x_2$ for all $a \in Q'_1$. Define the representation $\mathcal{R}'$ of Q' by restricting the representation $\mathcal{R}$, and repeat the process described above.

If $\mathcal{R}$ is a non-trivial representation of Q, we will eventually find $x_n \in Q_0$ such that $V_{x_n} \neq \{0\}$ but $V_{h(a)} = \{0\}$ for all $a \in Q_1$ such that $t(a) = x_n$.

Construct a representation $\mathcal{S}$ of Q by taking

$$\mathcal{S} = (\{W_i = \delta_{ni}\mathbf{k}\}, \{\phi_a = 0 \text{ for all } a \in Q_1\}).$$

Then $\mathcal{S}$ is a proper canonical subrepresentation of $\mathcal{R}$. To see this, observe that $W_i \subseteq V_i$ for all $i \in Q_0$ and define the inclusion morphism from $\mathcal{S}$ into $\mathcal{R}$ by $P = \{P_i : W_i \hookrightarrow V_i | i \in Q_0\}$.

To check that all maps commute, first note that for $a \in Q_1$ such that $t(a) \neq x$, $W_{t(a)} = \{0\}$. So $\psi_a : W_{t(a)} \to W_{h(a)}$ and $P_{t(a)} : W_{t(a)} \to V_{t(a)}$ must both be the zero map. Hence, for all $a \in Q_1$ such that $t(a) \neq x$ we have: $P_{h(a)} \circ \psi_a = \varphi_a \circ P_{t(a)} = 0$ so the morphism commutes.

Now, for all a such that $t(a) = x$, we know that $V_{h(a)} = W_{h(a)} = \{0\}$. So $\varphi_a : V_{t(a)} \to V_{h(a)}$, $\psi_a : W_{t(a)} \to W_{h(a)}$ and $P_{h(a)} : W_{h(a)} \to V_{h(a)}$ must all be the zero map. So for all $a \in Q_1$ such that $t(a) = x$, we have $P_{h(a)} \circ \psi_a = \varphi_a \circ P_{t(a)} = 0$ and the morphism commutes.

Therefore, $\mathcal{S}$ is a subrepresentation of $\mathcal{R}$ of the desired form. □

3.2.4 Indecomposable Representations

Here we will work with the examples we have given above. The invertibility of maps and the dimension vectors will play an important role in giving all the indecomposable representations for some given quiver.

Example 3.2.5. A representation $\mathcal{R}$ of the Jordan quiver J is indecomposable if it is isomorphic to a representation with matrix for ϕ_a in Jordan form with a single eigenblock, as such a matrix cannot be rewritten as a direct sum of two smaller matrices.

Example 3.2.6. For the oriented 2-Kronecker quiver C_2 in Example 3.1.2, we have the following classification:

Proposition 5. *A representation $\mathcal{R} = (\{V_1 = \mathbb{C}^m, V_2 = \mathbb{C}^n\}, \{\phi_a, \phi_b\})$ of C_2 is indecomposable if and only if one of the following holds:*

- *$\mathcal{R} \cong \mathcal{R}' = (\{V_1, V_2\}, \{Id, J_\lambda\})$ where J_λ is a matrix in Jordan normal form with only one eigenblock.*
- *$(\phi_b \circ \phi_a)^k = 0$ for some $k \in \mathbb{Z}^+$ and $\dim \ker \phi_b \circ \phi_a = 1$.*

Proof. Without loss of generality, $m \geq n$.

We describe the possible cases, and prove decomposability or indecomposability for each case.

1. If the composite map $\phi_b \circ \phi_a : V_1 \to V_1$ is invertible, then we must have $m = n$, and ϕ_a, ϕ_b both invertible. Thus, by changing bases, we can find an isomorphic representation with $\phi'_a = I$, ϕ'_b represented by a matrix in Jordan form.

   Then, as shown above, $\mathcal{R}$ is indecomposable if and only if the matrix for ϕ'_b has only one Jordan block.

2. If the composite map $\phi_b \circ \phi_a : V_1 \to V_1$ is not invertible, we have two cases:

   (a) $\phi_b \circ \phi_a$ is nilpotent, i.e. $(\phi_b \circ \phi_a)^k = 0$ for some $k \in \mathbb{Z}^+$

   i. Suppose $\dim \ker \phi_b \circ \phi_a = 1$. Then take $x \in \ker \phi_b \circ \phi_a$. Then any y in the kernel of $\phi_b \circ \phi_a$ must be a scalar multiple of x. Suppose $\mathcal{R}$ is not indecomposable, so that $\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}''$ where

   $$\mathcal{R}' = (\{W_1, W_2\}, \{A|_{W_1}, B|_{W_2}\})\ ,\ \mathcal{R}'' = (\{U_1, U_2\}, \{A|_{U_1}, B|_{U_2}\})$$

   are both non-trivial. Without loss of generality, $x \in W_1$.

   First suppose $y \in U_1, y \neq 0$. Then by definition of decomposability, $(\phi_b \circ \phi_a)^i \in U_1$ for all $i \in \mathbb{Z}_+$. But $(\phi_b \circ \phi_a)^k = 0$, so pick the least $j \in \mathbb{Z}^+$ such that $(\phi_b \circ \phi_a)^j = 0$. Then $(\phi_b \circ \phi_a)^{j-1} \in \ker \phi_b \circ \phi_a$ so for the y chosen above, $(\phi_b \circ \phi_a)^{j-1} y = \lambda x \in W_1$. But by assumption, $y \in U_1$. Thus, $U_1 = \{0\}$ so $V_1 = W_1$.

   Now suppose $y \in U_2,\ y \neq 0$. Then, by a dimension counting argument, either $y = \phi_a x$ for some $x \in V_1$ or $\phi_b y = x$ for some nonzero $x \in V_1$. In either case, $y \in W_2$ by the invariance of subrepresentations. But by assumption, $y \in U_2$, so $U_2 = \{0\}$ so $V_2 = W_2$. Therefore, $\mathcal{R}''$ in the direct sum is the trivial subrepresentation, so $\mathcal{R}$ is indecomposable.

   ii. Suppose $\dim \ker \phi_b \circ \phi_a > 1$. Write $\ker \phi_b \circ \phi_a = W_0 \oplus U_0$, both of which are non-zero. For $x \in V_1$, write $j_x \in \mathbb{Z}^+$ is the minimal integer such that $(BA)^{j_x} = 0$, and define:

   $$W_1 = \{x \in V_1 | (\phi_b \circ \phi_a) j_x - 1x \in W_0\}\ ,\ U_1 = \{x \in V_1 | (\phi_b \circ \phi_a)^{j_x - 1} \in U_0)\}$$

   These two sets define a decomposition of $\mathcal{R}$, so $\mathcal{R}$ is decomposable.

   (b) $\phi_b \circ \phi_a$ is not nilpotent; i.e. $(\phi_b \circ \phi_a)^k \neq 0$ for all $k \in \mathbb{Z}^+$. Then there is some integer j such that $V_1 = \ker(\phi_b \circ \phi_a)^j \oplus W_1$ and $(\phi_b \circ \phi_a)^j|_{W_1}$ is invertible. These sets define a decomposition of $\mathcal{R}$, so $\mathcal{R}$ is decomposable. ☐

Corollary 6. *The vector spaces* V_1, V_2 *of an indecomposable representation of* C_2 *can only have dimensions* $\dim V_1 = m = n = \dim V_2$ *or* $\dim V_1 = m$*, where* $m \pm 1 = n = \dim V_2$.

Proof. In the first case, V_1 and V_2 must be equidimensional. In the second case, $\dim \ker \phi_b \circ \phi_a = 1$ implies $|\dim V_1 - \dim V_2| \leq 1$. ☐

3.3 Lie Algebras and Their Representations

Definition 7. A **Lie Algebra** $\mathfrak{g}$ is a (non-associative) algebra with the multiplication rule given by a bilinear map $[\ ,\ ]$ which satisfies

- $[x, x] = 0$ for all $x \in \mathfrak{g}$,
- $[x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

These two properties imply that the [,] operation is anti-symmetric, i.e. $[x,y] = -[y,x]$ for all $x, y \in \mathfrak{g}$.

We can construct a Lie algebra from any associative algebra by defining the bracket operation as the commutator $[a,b] = ab - ba$.

3.3.1 Representations of $\mathfrak{sl}_2(\mathbf{k})$

The simple linear algebra $\mathfrak{sl}_2(\mathbf{k})$= $\{A \in M_2(\mathbf{k}) | \operatorname{tr} A = 0)\}$ of traceless 2×2 matrices is a Lie Algebra with bracket operation defined by the commutator $[A, B] = AB - BA$ and basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will describe the isomorphism classes of certain subclasses of the simple and indecomposable representations of $\mathfrak{sl}_2(\mathbf{k})$ and show that these correspond to simple and indecomposable representations of the 2-Kroneker quiver with an oriented cycle under the relation $ab = 0$, as described in Example 3.1.5.

We will restrict ourselves to the category $\mathcal{O}(\mathfrak{sl}_2)$ of representations of $\mathfrak{sl}_2(\mathbf{k})$ such that

- $V = \bigoplus_{k \in Z} V_k$, where $V_k = \{v \in V | hv = kv\}$ is the eigenspace with eigenvalue k for the action of h on V,
- $V_k = 0$ for $k \geq 0$,
- Each V_k is finite dimensional.

Given $v \in V_k$, using the bracket properties, calculation gives:

- $h(v) = kv$,
- $h(f(v)) = (k-2)f(v)$,
- $h(e(v)) = (k+2)e(v)$.

In other words, the action of f takes the eigenspace V_k with eigenvalue k to V_{k-2} with eigenvalue $k-2$, and the action of e takes V_k to V_{k+2}.

By the given properties, we have that each representation has a maximal eigenvalue $m \in \mathbb{Z}$ and:

$$V_k = \begin{cases} \mathbf{k} & \text{if } k = m - 2i \text{ for } i \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we take $v_0 \in V_m$, the set $\mathcal{B}$=$\{v_i | i \in \mathbb{Z}^+\}$, where $v_i = f^i(v_0)$, defines a basis for

$$V(m) = \bigoplus_{k \leq m} V_k.$$

From the equations above, we can calculate that $e(v_i) = i(m - i + 1)v_{i+1}$.

Now, we will describe a chain of examples of such representations; for this we assume that $\mathbf{k} = \mathbb{C}$.

Example 3.3.1. (**The Verma Module**)

Let $M(m)$ be the $\mathfrak{sl}_2$-module with underlying vector space

$$M(m) = \bigoplus_{i \geq 0} \mathbf{k} v_i$$

and the action given by

$$h(v_0) = mv_0, \qquad v_i = f^i(v_0), \qquad e(v_i) = i(m-i+1)v_{i+1}.$$

It is easy to check that this is in fact a representation and we have a diagramatic picture as in Figure 3.1.

We have defined m as the greatest eigenvalue of $M(m)$. If m is negative, then the map e does not annihilate any of the other eigenspaces, and we have an infinite-dimensional simple representation.

If the greatest eigenvalue, m, is nonnegative, the action of e will annihilate the eigenspace $M(m)_{-m-2}$ since $e(v_{m+1}) = 0$. In this case, the representation will not be simple; in fact, the $\mathbf{k}$-subspace

$$\bigoplus_{j \geq m+1} \mathbf{k} v_j$$

is a subrepresentation isomorphic to $M(-m-2)$. It will, however, be indecomposable, because the subspace

$$\bigoplus_{-m \leq i \leq m} M(m)_i$$

is not invariant under the action of f.

Now, for $m \geq 0$ $m \in \mathbb{Z}$, taking the quotient representation

$$V(m) = M(m)/M(-m-2)$$

gives a second example of simple representation, the only one with nonnegative integer maximal eigenvalue, $V(m)$. Its structure is shown in Figure 3.2(a).

These representations can be related by the following non-split short exact sequence:

$$0 \to V(-m-2) \to M(m) \to V(m) \to 0.$$

Example 3.3.2. ($P(-m-2)$ and $M^*(m)$)

Let $M(m)$ be the Verma module as defined above, and define another linearly independent eigenvector w_0 with eigenvalue $-m-2$ such that $e(w_0) = v_m$. From w_0, we can derive another set $\{w_i\}_{i \in \mathbb{N}}$ of eigenvectors by the rule $w_i = f^i(w_0)$ for each eigenvalue $\lambda = -m-2(i+1)$. Take the direct sum of the Verma Module with the eigenspaces spanned by these w_i's together with the action of e given by $e(w_i) = i(-m-i-1)w_{i-1} + v_{m+i}$.

Now we can consider the $\mathbf{k}$-vector space

$$M(m) \oplus \bigoplus_{j \geq 0} \mathbf{k} w_j$$

with action given by $h(w_0) = (-m-2)w_0$, $w_i = f^i(w_0)$ and $e(w_i) = i(-m-i-1)w_{i-1} + v_{m+i}$. It is easy to verify that these formulas turn the vector space $\bigoplus_{j \geq 0} \mathbf{k} v_j \oplus \bigoplus_{i \geq 0} \mathbf{k} w_i$ into a module belonging to category $\mathcal{O}(\mathfrak{sl}_2)$. We denote this module by $P(-m-2)$ and it has the diagram shown in Figure 3.2(b).

Notice that the $\mathbf{k}$-subspace $\bigoplus_{j \geq m+1} \mathbf{k} v_j$ is a subrepresentation isomorphic to $M(-m-2)$. Then, taking the quotient of these two representations, we define $M^*(m) = P(-m-2)/M(-m-2)$. This gives another non-split short exact sequence:

$$0 \to M(-m-2) \to P(-m-2) \to M^*(m) \to 0.$$

Furthermore, $M^*(m)$ has the diagram shown in Figure 3.2(c).

$$\begin{tikzcd}
\mathbf{k}v_0 \arrow[d, shift right, "f"'] \\
\vdots \arrow[u, shift right, "e"'] \arrow[d, shift right, "f"'] \\
\mathbf{k}v_m \arrow[u, shift right, "e"'] \arrow[d, "f"'] \\
\mathbf{k}v_{m+1} \arrow[d, shift right, "f"'] \\
\vdots \arrow[u, shift right, "e"']
\end{tikzcd}$$

Figure 3.1: The Verma module $M(m)$.

Also, we can see that $M^*(m)$ has $V(m)$ as a subrepresentation which gives directly the next non-split short exact sequence:

$$0 \to V(m) \to M^*(m) \to M(-m-2) \to 0.$$

Finally, as $P(-m-2)$ has $M(m)$ as a subrepresentation, we get another short exact sequence:

$$0 \to M(m) \to P(-m-2) \to M(-m-2) \to 0.$$

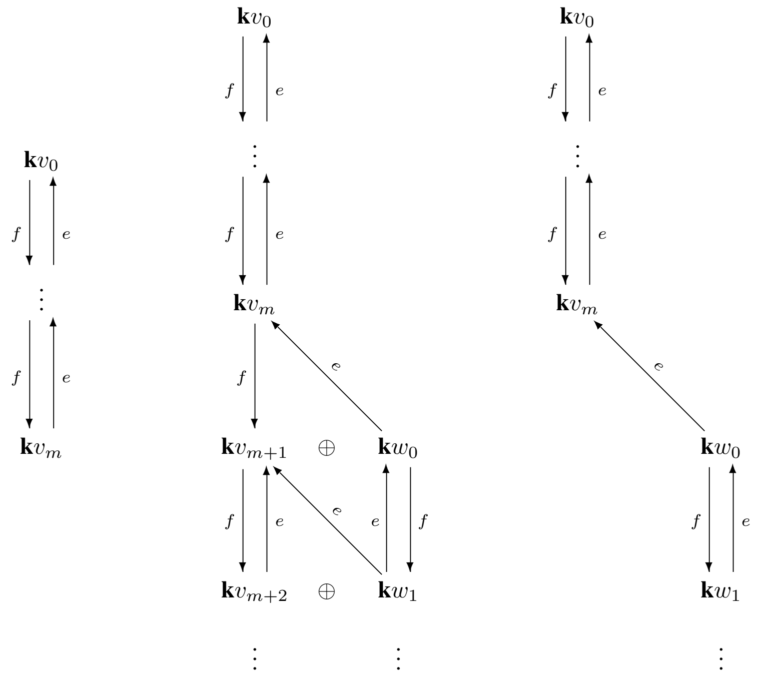
The following proposition tell us that the above examples are actually all of the examples of indecomposable modules in $\mathcal{O}(\mathfrak{sl}_2)$.

Proposition 8. *The following short exact sequences*

$$\begin{gathered}
0 \to M(-m-2) \to M(m) \to V(m) \to 0 \\
0 \to V(m) \to M^*(m) \to M(-m-2) \to 0 \\
0 \to M(-m-2) \to P(-m-2) \to M^*(m) \to 0 \\
0 \to M(m) \to P(-m-2) \to M(-m-2) \to 0
\end{gathered}$$

are a complete set of equivalence class representatives of non-split short exact sequences of representations in the category $\mathcal{O}(\mathfrak{sl}_2)$.

A proof can be found in [FH]. In particular, every indecomposable representation in the category $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to one of the examples given above.



(a) $V(m)$ (b) $P(-m-2)$ (c) $M^*(m)$

Figure 3.2: More $\mathfrak{sl}_2$-modules.

3.4 A matching example

At this point, for each nonnegative integer m, we have found two simple representations ($M(-m-2)$ and $V(m)$) and three indecomposable representations ($M(m)$, $M^*(m)$, $P(-m-2)$) of $\mathfrak{sl}_2(\mathbf{k})$.

The next step is to match these representations with representations of some quiver. In order to do that, we consider the quiver with relation given in Example 3.1.5. The following proposition gives the classification of the simple and indecomposable representations of this quiver.

Proposition 9. *For the quiver with relation given in Example 3.1.5, there are two simple representations given by*

1.

$$\begin{tikzcd}
\mathbf{k}\bullet \arrow[r, shift right, "0"'] & \bullet 0 \arrow[l, shift right, "0"']
\end{tikzcd}$$

2.

$$\begin{tikzcd}
0\bullet \arrow[r, shift right, "0"'] & \bullet \mathbf{k} \arrow[l, shift right, "0"']
\end{tikzcd}$$

and there are three indecomposable representations given by:

3.

$$\begin{tikzcd}\mathbf{k}\bullet \arrow[r, shift right, "0"'] & \bullet\mathbf{k} \arrow[l, shift right, "\mathbf{1}"']\end{tikzcd}$$

4.

$$\begin{tikzcd}\mathbf{k}\bullet \arrow[r, shift right, "\mathbf{1}"'] & \bullet\mathbf{k} \arrow[l, shift right, "0"']\end{tikzcd}$$

5.

$$\begin{tikzcd}\mathbf{k}\bullet \arrow[r, shift right, "\binom{0}{1}"'] & \bullet\mathbf{k}^2 \arrow[l, shift right, "(0\ 1)"']\end{tikzcd}$$

To prove this proposition, we use the previous results described above.

Notice that we have exactly the same number of simple and indecomposable representation. This suggests the following correspondence between each simple and indecomposable representations:

Proposition 10. *There is a bijective correspondence, which preserves inclusions and quotients, between equivalence classes of simple and indecomposable modules of the 2-Kronecker quiver with relation from Example 3.1.5 and the simple and indecomposable modules of $\mathcal{O}(\mathfrak{sl}_2)$, given as follows:*

- $1 \leftrightarrow V(m)$
- $2 \leftrightarrow M(-m-2)$
- $3 \leftrightarrow M^*(m)$
- $4 \leftrightarrow M(m)$
- $5 \leftrightarrow P(-m-2)$

Proof. By Proposition 8, we have short exact sequences

$$\begin{gathered}0 \to V(m) \to M^*(m) \to M(-m-2) \to 0,\\ 0 \to M(-m-2) \to M(m) \to V(m) \to 0,\\ 0 \to M(-m-2) \to P(-m-2) \to M^*(m) \to 0,\\ 0 \to M(m) \to P(-m-2) \to M(-m-2) \to 0.\end{gathered}$$

Quiver representation #5 above has representations #2 and #3 as subrepresentations; #4 has #2 as a subrepresentation; and #3 has #1 as subrepresentation.

The result follows by observing that $P(-m-2)$ has two lie algebra subrepresentations corresponding to the two quiver subrepresentations of #5. Then by a dimension analysis for the last exact sequence, we get the correspondence between the two simple representations. Similarly, $M(m)$ has $M(-m-2)$ as a subrepresentation, corresponding to the quiver subrepresentation #2 in #4, and $M^*(m)$ has $V(m)$ as a subrepresentation, corresponding to the quiver subrepresentation #1 in #3. ☐

This correspondence is not an accident. In fact, the 2-Kronecker quiver corresponds to the Lie algebra $\mathfrak{sl}_2$ under a correspondence developed by Kac and Moody. In this more general matching, a quiver corresponds to a matrix representing (t_{ij}) where t_{ij} is the number of arrows between vertices i and j. This matrix then is used to formulate a set of relations which describe the corresponding Lie algebra.

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References

[De] Harm Derksen: Lecture notes for Math 711, Winter 2001. `http://www.math.lsa.umich.edu/~hderksen/math711.w01/math711.html`

[FS] Jürgen Fuchs and Christoph Schweigert: *Symmetries, Lie Algebras and Representations: A Graduate Course for Physics*. Cambridge: Cambridge Univ. Press, 1997.

[FH] William Fulton, and Joe Harris: *Representation Theory: a First Course*. New York: Springer, 1991 (Graduate Texts in Math. **129**).

[Ha] Paul R. Halmos: *Finite-Dimensional Vector Spaces*, 2nd ed. Princeton, N.J.: Van Nostrand, 1958.

[Hu] James E. Humphreys: *Introduction to Lie Algebras and Representation Theory*. New York: Springer, 1987 (Graduate Texts in Math. **9**).

[Ka] Victor G. Kac: *Infinite Dimensional Lie Algebras*, 3rd ed. Cambridge: Cambridge Univ. Press, 1990.

[Sa] Alistair Savage: Finite-dimensional algebras and quivers, preprint. `arXiv:math/0505082v1` [math.RA].