STUDENT ARTICLE

5 Problems of Circle Tangency

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Abstract

This article presents a (very) brief overview of geometric problems involving tangent circles. In addition to defining the technique of inversion, we give two example problems with full solutions and suggest another challenge problem related to Pappus circles.

5.1 Introduction

Geometric problems dealing with tangent circles have a long history and arise in surprising places. For example, the broad challenge "given three objects (where an object may be a line, circle, or point), draw a circle which is tangent to each" is known as **Apollonius' Problem** after the 3rd century BCE Greek geometer Apollonius, who wrote two works considering the problem. The **Descartes circle theorem** is a special case (the hardest special case) of Apollonius' Problem [Cox]. During Japan's isolationist period between the mid 17th and 19th centuries, inscribed geometry problems known as **sangaku** which often dealt with circle tangency were hung from religious buildings [RF]. In more modern use, a particular set of circles with rational centers known as **Ford circles** may be used to prove the **Hurwitz theorem**:

Theorem 1 (Hurwitz Theorem)**.** *If $k \geq 1/\sqrt{5}$, then for each irrational number w there are infinitely many fractions p/q satisfying*

$$\left|\frac{p}{q} - w\right| < \frac{k}{q^2}. \tag{5.1}$$

If $k < 1/\sqrt{5}$, then there exist irrationals w for which (5.1) has only finitely many solutions $\frac{p}{q} \in \mathbb{Q}$.

For more information on Ford circles, the reader is referred to L.R. Ford's original article [Fo].

Instead of providing a full historical overview of the subject or presenting new connections, this article demonstrates two particular problems of circle tangency and solves them using two very different strategies. Along the way, we will encounter **inversions of the plane**, an operation in planar geometry which is interesting but often overlooked. The paper is divided into three sections. First, we present and solve one problem without inversion. We then define inversions and list their important properties. Finally, we present another problem and solve it using inversion. The author hopes these two examples will help convince the reader of the beauty of the subject.

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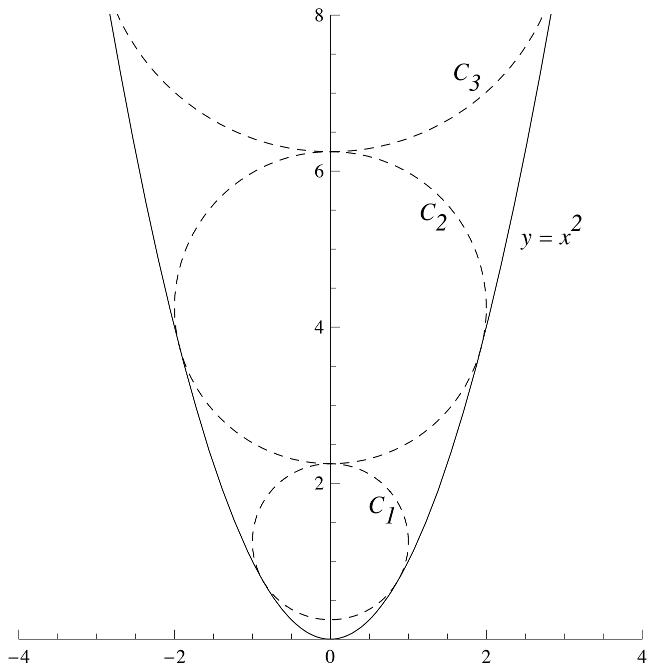

Figure 5.1: Mutually Tangent Circles Inscribed in a Parabola.

5.2 Circles in the Parabola

This section is dedicated to a solution of the following problem. To the author's knowledge, this problem was first printed as Exercise 63/64 in Section 7.1 of [SM].

Problem 1. *Let P be the parabola $y = x^2$, and let C_1 be the circle of radius 1 which is tangent to P at two points. Iteratively define a sequence of circles $\{C_2, C_3, \dots\}$, where each C_n is located above and tangent at a point to C_{n-1} and tangent at two points to P. Find the radius of C_n.*

Solution. We present an analytic solution which relies only on high school algebra techniques. Let C_n be the circle centered at (x_n, y_n) and having radius r_n. Notice that by symmetry, we must have $x_n = 0$ for C_n to be tangent to P at two points. Now the curves C_n, P are described by the equations

$$x^2 + (y - y_n)^2 = r_n^2, \tag{5.2}$$

$$y = x^2. \tag{5.3}$$

Substituting (5.3) into (5.2) gives the intersection condition

$$y + (y - y_n)^2 = r_n^2$$

$$y^2 + (1 - 2y_n)y + (y_n^2 - r_n^2) = 0. \tag{5.4}$$

Tangency of the curves means that the intersection equation (5.4) must have a double root. This implies the discriminant is zero:

$$0 = (1-2y_n)^2 - 4(1)(y_n^2 - r_n^2) = 1 - 4y_n + 4r_n^2,$$

so $y_n = \frac{1}{4} + r_n^2$.

Since C_n and C_{n-1} are tangent, the sum of their radii equals the distance between their centers. Writing this equation and substituting the above relation between y_n and r_n, we get

$$\begin{aligned} r_n + r_{n-1} &= y_n - y_{n-1} \\ r_n + r_{n-1} &= \left(\frac{1}{4} + r_n^2\right) - \left(\frac{1}{4} + r_{n-1}^2\right) \\ r_n + r_{n-1} &= r_n^2 - r_{n-1}^2 \\ 1 &= r_n - r_{n-1}. \end{aligned}$$

Thus, since $r_n = r_{n-1} + 1$ and $r_1 = 1$, the radius r_n of the nth circle is n. □

5.3 Inversions in the Plane

Inversion in the plane is a geometric "reflection" technique that uses a circle in lieu of a linear axis of symmetry. The formal definition of such a transformation is

Definition 2. Given a circle C with center O and radius r, the **inversion about** C is the map taking each point P to the point on ray $\overrightarrow{OP}$ of distance $r^2/|OP|$ from O.

Notice that inversion about the circle C fixes all the points of C and exchanges the groups of points that lie inside and outside the circle. In particular, the center O is taken to infinity. Because it is not well-defined at the center of the circle, inversion should really be viewed as a map on the **one-point compactification** of the plane formed by adding a point at infinity. This level of detail is not important for our purposes, but we shall speak occasionally about the point at infinity.

Notice that we obtain the identity transformation if we perform two inversions about the same circle; thus inversions are bijections of the (compactification of the) plane. Furthermore, though we will only employ inversions of the plane $\mathbb{R}^2$, it is worth noting that inversions may be defined naturally in higher-dimensional space $\mathbb{R}^n$, where the object to be inverted about is an $(n-1)$-dimensional scaling and translate of the sphere S^{n-1}. If we consider the inversions about two different circles centered at the same point, then the inversions are dilations of each other. Thus, the radius is not as interesting as the center; we often refer to inversion about a point P as shorthand for inversion about the circle of radius 1 centered at P.

The following result gives the primary use of inversions for circle problems:

Theorem 3. *Any inversion in the plane maps circles to circles, where lines are viewed as circles through infinity.*

Thus, since inversions are bijections which preserve the class of circles, they map a set of tangent circles to another set of tangent circles. Clever choice of the center for inversion could help make this new set particularly easy to work with.

The interested reader is referred to [Coo] for a more in-depth discussion of inversion, as well as a proof of Theorem 3.

5.4 Fibonacci Circles

This section is dedicated to an inversion solution to the following problem. This problem may also be found as Exercise 61/62 in Section 7.1 of [SM]; however, it is not original to this text.

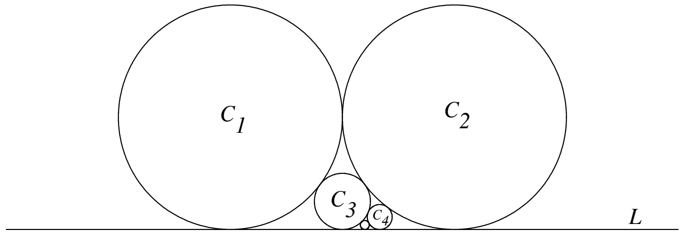

Figure 5.2: Fibonacci Circles.

Problem 2. *Let L be a line, and let C_1, C_2 be two circles of radius 1 tangent to each other and both tangent to L. Define a sequence of circles $C_3, C_4, \dots$ where C_n is tangent to C_{n-1}, C_{n-2}, and L. Find the radius of C_n.*

Solution. Label by r_n the radius of C_n; thus $r_1 = r_2 = 1$. Consider now C_n, $n \geq 3$, and let P be the point at which C_{n-1} intersects C_{n-2}. Let I be the inversion of the plane about P. Since inversions are bijections taking circles to circles, C_{n-1} and C_{n-2} must each map to circles which only intersect at $I(P)$, the point at infinity. Thus, $I(C_{n-1})$ and $I(C_{n-2})$ are parallel lines.

The minimum distance from P to $I(C_{n-1})$ is the inverse of the maximum distance from P to C_{n-1}, which is $2r_{n-1}$. Similarly, the maximum distance from P to C_{n-2} is $2r_{n-2}$, so $I(C_{n-1})$ and $I(C_{n-2})$ are parallel lines with separation $(2r_{n-1})^{-1} + (2r_{n-2})^{-1}$.

Since L is a line (circle through infinity) which is tangent to C_{n-1} and C_{n-2}, $I(L)$ is a circle through P tangent to the lines $I(C_{n-1})$ and $I(C_{n-2})$. Further, since C_n is the circle (not passing through P) tangent to C_{n-1}, C_{n-2}, and L, $I(C_n)$ must be a circle, not passing through P, tangent to the lines $I(C_{n-1})$ and $I(C_{n-2})$ as well as to the circle $I(L)$. This fully determines the situation, which is shown in the figure below.

Note that any circles tangent to both $I(C_{n-1})$ and $I(C_{n-2})$ must fit in between the lines and have diameter given by the line separation, $(2r_{n-1})^{-1} + (2r_{n-2})^{-1}$. Let $\ell = \frac{1}{2}(2r_{n-1})^{-1} + \frac{1}{2}(2r_{n-2})^{-1}$ be the associated radius.

Denote by B the line through the centers of $I(L)$ and $I(C_n)$; B is the line halfway between $I(C_{n-1})$ and $I(C_{n-2})$. Note that P will be at least as close to $I(C_{n-1})$ as to $I(C_{n-2})$, since we expect $r_{n-1} \leq r_{n-2}$ by geometric intuition; this can be proven by induction along with the following calculations. Now P has distance $d_1 = (2r_{n-1})^{-1}$ from $I(C_{n-1})$, so the distance from P to line B is $\ell - d_1$. Let the closest point on B to P be D, that is, let D be the intersection of B with the perpendicular to B through P. Using the Pythagorean theorem, we can see that the distance from the center O of $I(L)$ to D is $\sqrt{\ell^2 - (\ell - d_1)^2} = \sqrt{2\ell d_1 - d_1^2}$. Now the distance between O and the center O' of $I(C_n)$ is 2ℓ, and this distance is along B, so the distance from D to O' is $\sqrt{2\ell d_1 - d_1^2} + 2\ell$. Finally, using the Pythagorean theorem again, we can see that the distance from P to O' is

$$|PO'| = \sqrt{(\ell - d_1)^2 + \left(\sqrt{2\ell d_1 - d_1^2} + 2\ell\right)^2} = \sqrt{5\ell^2 + 4\ell\sqrt{2\ell d_1 - d_1^2}}.$$

Since O' is the center of $I(C_n)$, a circle of radius ℓ, the least distance from P to $I(C_n)$ is $|PO'| - \ell$ and the maximum distance is $|PO'| + \ell$. If we perform another inversion about P, we map $I(C_n)$ back to C_n. The inversions of the previous minimum and maximum distance from P to $I(C_n)$ will give, respectively, the maximum and minimum distances from P to C_n; the difference between these is the diameter of C_n.

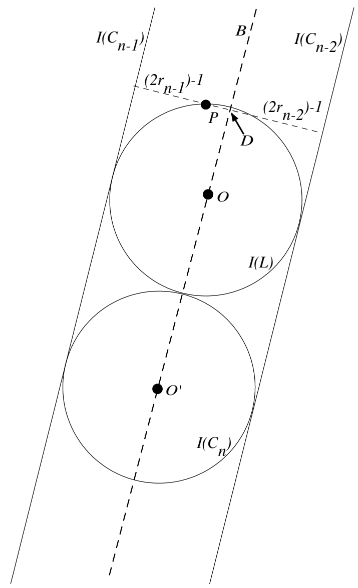

Figure 5.3: Inverted Fibonacci Circles.

Thus, we see that

$$\begin{aligned} 2r_n &= \frac{1}{|PO'| - \ell} - \frac{1}{|PO'| + \ell} \\ &= \frac{(|PO'| + \ell) - (|PO'| - \ell)}{|PO'|^2 - \ell^2} \\ &= \frac{2\ell}{4\ell^2 + 4\ell\sqrt{2\ell d_1 - d_1^2}}. \end{aligned} \tag{5.5}$$

After cancelling a 2ℓ in (5.5) and making the substitutions $d_1 = (2r_{n-1})^{-1}$ and $\ell = \frac{1}{2}(2r_{n-1})^{-1} + \frac{1}{2}(2r_{n-2})^{-1}$, we obtain

$$\begin{aligned} 2r_n &= \frac{1/2}{\frac{1}{2}(2r_{n-1})^{-1} + \frac{1}{2}(2r_{n-2})^{-1} + \sqrt{((2r_{n-1})^{-1} + (2r_{n-2})^{-1})(2r_{n-1})^{-1} - (2r_{n-1})^{-2}}} \\ 4r_n &= \frac{r_{n-1}r_{n-2}}{r_{n-2}/4 + r_{n-1}/4 + \sqrt{r_{n-1}r_{n-2}/4}} \\ r_n &= \frac{r_{n-1}r_{n-2}}{r_{n-1} + 2\sqrt{r_{n-1}r_{n-2}} + r_{n-2}} \\ r_n &= \frac{r_{n-1}r_{n-2}}{(\sqrt{r_{n-1}} + \sqrt{r_{n-2}})^2}. \end{aligned} \tag{5.6}$$

Taking an inverse and a square root of both sides of (5.6), we see that

$$\frac{1}{\sqrt{r_n}} = \frac{1}{\sqrt{r_{n-1}}} + \frac{1}{\sqrt{r_{n-2}}}, \tag{5.7}$$

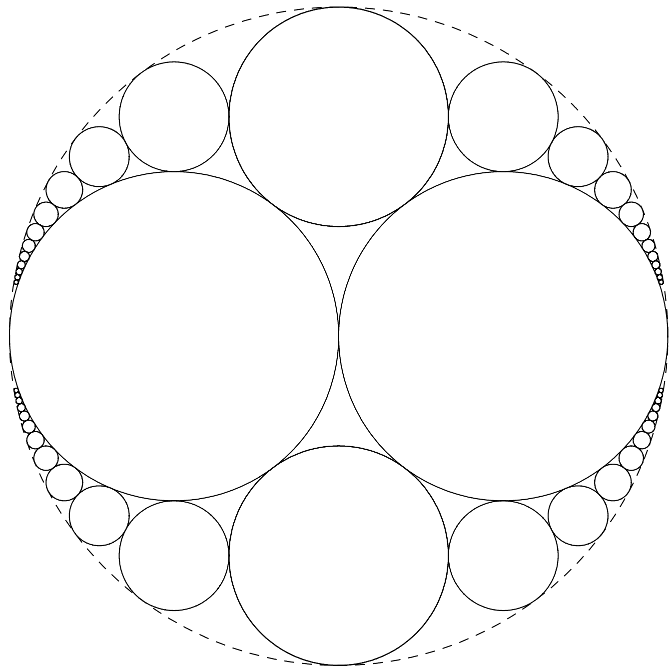

Figure 5.4: Multiple Pappus Chains.

which is the relation we were seeking. We can now complete the problem: let F_n be the nth Fibonacci number (indexed so that $F_1 = F_2 = 1$) and notice that $1/\sqrt{r_n}$ satisfies the Fibonacci relation. Thus, since $r_i = 1/F_i^2$ for $i = 1, 2$, this pattern holds in general; the radius of the nth circle is $1/F_n^2$. □

For the sake of honesty, we should observe that this relation may be derived more quickly by using analytic geometry techniques; inversion is not required. In fact, the relation (5.7) can be seen in a Japanese *sangaku* from 1824 [RF].

5.5 Conclusion

This paper has presented two examples which provide a taste for the variety of approaches in solving circle tangency problems. In particular, the author has found inversions of the plane to be extremely powerful. We close with a challenge: The reader is invited to find the total area of all the solid circles in Figure 5.4, below.

In Figure 5.4, the bounding, dotted circle has radius 2 and the two largest solid circles have radius 1. This problem is related to the so-called "Ancient Theorem" of Pappus, which has been studied in depth by Jakob Steiner [Coo]. Related constructions have also been found in Japanese *sangakus* [RF]. For more information and examples, we refer the reader to Martin Gardner's survey article [Ga].

5.6 Acknowledgment

The author wishes to dedicate this article to James Albrecht, in thanks both for posing the Fibonacci circles problem and in general for his camaraderie over the years.

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