FEATURE

9

APPLIED MATHEMATICS CORNER

Fireflies & Oscillators

Pablo Azar†
Harvard University '09
Cambridge, MA 02138
azar@fas.harvard.edu

9.1 Introduction

In 1990, R. Mirollo and S. Strogatz [MS] presented a coupled oscillator model explaining how synchronicity arises from self-organization. In this article, I will describe this model and its main result, as well as some applications to computer science. As the methods of pure mathematics become more and more complicated, it is very exciting to see how nature can still be explained and imitated using simple models.

9.2 Synchronization and Self-Organization

When ants forage for food and birds flock together, they form patterns that betray an ability to communicate. **Self-organization** is the idea that these patterns arise from local interactions (see [CDF]). Instead of following a master plan or being guided by a leader, individuals react to their local environments by following simple rules. The aggregation of these small actions is what leads to the final complicated pattern. Because fish look at their nearest neighbors to decide on their directions and velocities, they end up swimming in tight formations that are useful for evading predators. Ants decide where to forage for food by perceiving the local levels of pheromones deposited by other members of their colonies.

In this exposition, I will present a model of synchronous firefly flashing: males flash in unison to attract females. The purpose of this feature is to serve as an introduction to the study of self-organization and an invitation for the reader to investigate this topic further.

9.3 From Facts to Model

I will start by presenting some basic facts about fireflies and by showing a possible translation of these facts into a working mathematical model. An interesting first fact is the following:

Fact 1. *A firefly can produce periodic pulses of light.*

We can use this fact to start building the model. Let t denote time. Since the flashes are periodic, there must be a periodic function $f(t)$ with image $[0, 1]$ that measures how close the firefly is to flashing. When $f(t) = 1$ the firefly flashes and f falls back to 0. If the period of the firefly is ω then we have $f(t + \omega) = f(t)$. Though this model looks good, it has a fatal flaw: it does not explain synchronization. Two fireflies with the same ω will only synchronize if they start their cycles at the same moment, that is, only if they were synchronized from the beginning! To fix this, we must incorporate another fact into our model:

†Pablo Azar, Harvard '09, is an applied mathematics concentrator from Buenos Aires, Argentina. He is a founding member of The HCMR and currently serves on The HCMR's staff.

Fact 2. *A firefly can control the frequency of its pulses.*

To incorporate this fact, let $\phi(t)$ denote the time kept by the firefly's internal clock, expressed as a function of the time measured by objective clocks in the environment. The firefly flashes when $f(\phi(t)) = 1$, and f is periodic as a function of ϕ. This model also describes the behavior of a circuit that consists of a capacitor and a light bulb: $\phi(t)$ is the charge accumulated in the capacitor at time t, while $f(\phi(t))$ is the voltage. When the voltage reaches a certain threshold, the capacitor is discharged, illuminating the light bulb. A change in $\phi(t)$, which signifies a change in the speed at which the firefly thinks time is passing, is analogous to a change in the amount of energy it is going to use to charge the capacitor. There are two more things to be said about this analogy. The first is that as charge accumulates, voltage increases. Thus, f should be increasing as a function of ϕ. The second is that the first units of charge should have a larger impact than the succeeding units of charge, so that the rate of increase of f diminishes as f increases. It will not hurt to assume that f is smooth, so we can state these assumptions as $\frac{df}{d\phi} > 0$, $\frac{d^2 f}{d\phi^2} < 0$. These assumptions, of course, are only valid for within one cycle. They are not valid in the isolated moments of time when the firefly flashes and f drops immediately from 1 to 0.

The first model suggests that this charge always accumulates at a constant rate, but we want the firefly to be able to change this rate. The reason for this is

Fact 3. *A firefly responds to other fireflies.*

When one firefly sees other fireflies flash, it will want to accelerate the rate at which it accumulates charge, in order to bring itself closer to flashing.[1] Since the function f measures how close the firefly is to flashing, increasing f by a constant amount ϵ is a good response to a neighbor's flash. Of course, the firefly only controls the argument $\phi(t)$. This increase must be achieved by updating $\phi(t)$ to some $\phi'(t)$ for which $f(\phi'(t)) = f(\phi(t)) + \epsilon$. Note that, since f is concave, the change $\phi'(t) - \phi(t)$ must become larger as $f(t)$ approaches 1. If this boost brings the voltage to something larger than one, the firefly flashes, the potential is reset to zero, and the individual becomes synchronized with its neighbors.[2]

Why do we have this strange rule? Suppose that the firefly followed instead the policy of increasing the argument $\phi(t)$ always by a constant magnitude δ to $\phi'(t) = \phi(t) + \delta$. Then, because f is concave, the corresponding increases in f would get smaller as f is closer to 1. This is very inefficient if we want to achieve synchronicity quickly.

9.4 From Model to Facts

So far, our model is based on a few simple facts and some modeling assumptions. These assumptions are not carved in stone. For example, this model assumes that the effect of a firefly on its

[1] Why would a firefly want to do this? In this feature we explain how male fireflies synchronize to attract females. The question of why this would be evolutionary advantageous leads to an interesting intersection between game theory and biology. These models on evolutionary theory not only attempt to explain evolution, but also lend themselves to many applications such as optimization. If you are interested, some seminal references are Richard Dawkins' *The Selfish Gene* for the biology/game theory aspect and John H Holland's *Adaptation in Natural and Artificial Systems* for some non-biological applications.

[2] More formally, if we give each firefly an index i from 1 to n then:

$$f(\phi_i(t)) = 1 \implies \lim_{s \to t^+} f(\phi_j(s)) = \min(f(\phi_j(t)) + \epsilon, 1), \text{ for } i \neq j.$$

Note that what is hidden behind this equation is a change in $\phi_j(t)$. All fireflies share the same potential function f, but each can only accelerate its own internal, subjective time $\phi_j(t)$. The firefly must hence find a new value $\phi'_j(t)$ such that $f(\phi'_j(t)) = f(\phi_j(t)) + \epsilon$. Now, we had reasons to assume that f was strictly increasing. Hence, f has an inverse g and we can compute $\phi'_j(t) = g(\min(f(\phi_j(t)) + \epsilon))$. The reader who believes that we are assuming the problem away by introducing more notation should take relief by knowing that in applications, f is a familiar concave function like $\log(x)$, which has a familiar inverse e^x.

neighbors is a discrete boost, but other models work just as well assuming continuous boosts. Also, we have to watch out that this model does not lead to a dead end like the first one shown does. In fact, the model achieves what we want and more. First, it explains self-synchronization. Assume that we have n fireflies with initial charges $\phi_1(0), ..., \phi_n(0)$, such that each firefly boosts its neighbors' potentials by ϵ_i when it flashes. The values $(\phi_1(0), ..., \phi_n(0), \epsilon_1, ..., \epsilon_n)$ are the parameters of the model. Mirollo and Strogatz [MS] prove the following:

Theorem 4. *Let a model with n fireflies have parameters $\phi_1(0), ..., \phi_n(0), \epsilon_1, ..., \epsilon_n$. Assume that when a firefly flashes, it boosts all its neighbors. Then, for all initial parameters except those on a set of measure zero, there exists a time t_{conv} such that $f(\phi_1(t_{conv})) = f(\phi_2(t_{conv})) = ... = f(\phi_n(t_{conv}))$. At this moment, the fireflies are synchronized.*

This is equivalent to the following biological fact:

Fact 5. *If all fireflies in a group influence each other, they will eventually become synchronized.*

From a few simple facts about fireflies, one can build a mathematical model that explains how they achieve synchronization. But we can do even better. One of the most unrealistic assumptions in the Mirollo-Strogatz model is that one firefly influences all the others. However, self-organization relies on individuals reacting to their *local* environment. Fourteen years after Mirollo and Strogatz obtained Theorem 4, Lucarelli and Wang [LW] modified their model to obtain synchronization under the assumption that a firefly can only be influenced by its nearest neighbor.

There is one more thing that vouches for the Mirollo-Strogatz model of firefly synchronization: its simplicity and flexibility make it very applicable. The model can explain nature and can also imitate it. Based on the Mirollo-Strogatz model, Werner-Allen, Tewari, Patel, Welsh, and Nagpal [WTP] have developed the **Reachback Firefly Algorithm** to induce synchronicity in sensor networks.

The applications of this reasoning go far beyond fireflies. For example, suppose that a robot had an artificial eye with millions of tiny sensors. It would be inefficient for each sensor to send its information to the central processing unit at idiosyncratic intervals. We would rather have all sensors send their information simultaneously. Similarly, suppose that we had thousands of small, cheap robots exploring a foreign planet and controlled by a central base. We would want all robots to send their information simultaneously, especially if the central base had to make very quick decisions.

Finally, a word must be said about simplicity. What first attracted me to biologically inspired models was that they explained nature with new but simple concepts. Applied mathematics does not necessarily move forward by becoming more abstract and complicated. Sometimes, new developments arise with just simple math and common sense.

References

[CDF] Scott Camazine, Jean-Louis Deneubourg, Neigel R. Franks, James Sneyd, Guy Theraulaz, and Eric Bonabeau: *Self-Organization in Biological Systems*. Princeton University Press, 2003.

[LW] Dennis Lucarelli and I-Jeng Wang: Decentralized synchronization protocols with nearest neighbor communication. *Proceedings of the 2nd International Conference on Embedded Networked Sensor Systems* (2004), 62–68.

[MS] Renato E. Mirollo and Steven H. Strogatz: Synchronization of Pulse-Coupled Biological Oscillators, *SIAM J. on App. Math.* **50** #6 (1990), 1645–1662.

[WTP] Geoffrey Werner-Allen, Geetika Tewari, Ankit Patel, Matt Welsh, and Radhika Nagpal: Firefly-inspired sensor network synchronicity with realistic radio effects. *Proceedings of the Third International Conference on Embedded Networked Sensor Systems* (2005), 142–153.