STUDENT ARTICLE

1 Determining the Genus of a Graph

Andres Perez[†]
Harvey Mudd College '09
Claremont, CA 91711
aperez@hmc.edu

Abstract

This paper investigates an important aspect of topological graph theory: methods for determining the genus of a graph. We discuss the classification of higher-order surfaces and then determine bounds on the genera of graphs embedded in orientable surfaces. After generalizing Euler's Formula to include graphs embedded on these surfaces, we derive upper and lower bounds for the genera of various families of simple graphs. We then examine some formulas for the genera of particular graphs.

1.1 Introduction

A graph G is **planar** if and only if it can be drawn in the plane such that none of its edges cross. Two examples of non-planar graphs are the complete graph on five vertices K_5 and the complete bipartite graph $K_{3,3}$. A **complete graph** consists of a set of completely connected vertices; a **complete bipartite graph** consists of two independent sets of vertices in which all the vertices in one group are connected to all the vertices in the other group. In 1930, Kazimierz Kuratowski [Ku] arrived at a result that is now known as **Kuratowski's Theorem**:

Theorem 1 (Kuratowski's Theorem)**.** *A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.*

This characterization of planar graphs leads to a natural question: Does there exist a surface upon which these non-planar graphs can be embedded such that there are no edge crossings? From a topological viewpoint, drawing a graph on a flat plane is equivalent to drawing the same graph on a sphere. We can verify this by taking the stereographic projection of a sphere, i.e. we can unravel the surface of a sphere by creating a hole at the north pole and stretching the sphere's surface out onto the plane (the pole of the sphere can be placed inside some region of the graph and this becomes the "outer face" of the graph). Thus, a graph is planar if and only if it can be drawn on the sphere in a way such that no edges cross.

Supposing we had one edge crossing in a graph G, we could draw G without any edges crossing by introducing a "handle" to our sphere (which is the topological equivalent to the torus–see Figure 1.1), and drawing one of the edges over the handle so that it no longer crosses the other edge. In this way, we could properly embed the graph $K_{3,3}$ on the torus without any edges crossing (see Figure 1.2).

The torus is an example of a **surface of higher genus**. The sphere is designated to be the surface S_0; the surface formed by adding k handles to the sphere is denoted S_k. The torus is therefore S_1, the double-torus S_2, and so on, where the **genus** of S_k is the number of handles, k.

[†]Andres Perez, Harvey Mudd College '09, is a mathematics major living in Harvey Mudd's West Dorm in Claremont, CA. His family is originally from Peru and he was born in Lake Forest, IL, where he attended Lake Forest Academy. His mathematical interests include graph theory, combinatorics and algebra; he also appreciates theoretical computer science. Beyond math, his interests include poker, photography, and rock climbing.

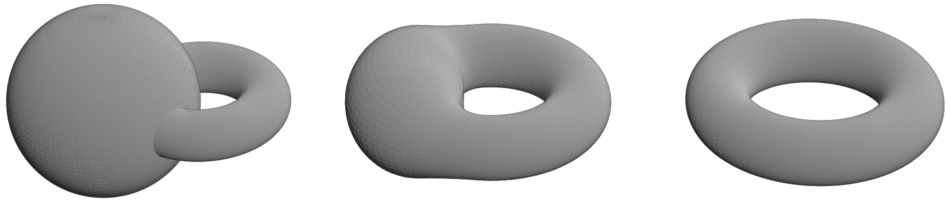


Figure 1.1: Introducing a handle to a sphere is topologically equivalent to a torus.

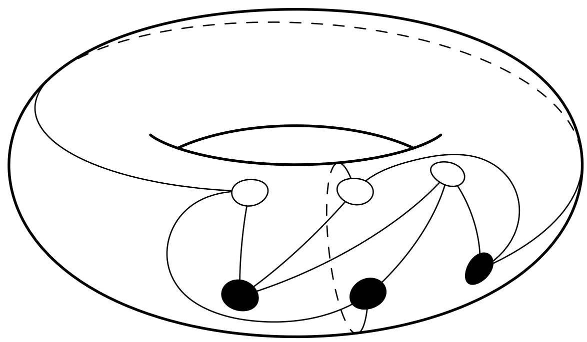


Figure 1.2: The graph $K_{3,3}$ drawn without edge crossings on the torus.

The main purpose of this paper is to determine, for a graph G, the minimal k such that G can be drawn on S_k without edge crossings.

We see that one upper limit to this number is the number of crossings in a drawing of the graph on S_0: we simply introduce a handle for each instance of two edges crossing. But what is the lowest-genus surface required? We define this number to be the **genus** of a graph G (plural, **genera**), denoted $\gamma(G)$. Thus for any planar graph G, $\gamma(G)$ is zero. Both $K_{3,3}$ and K_5 are of genus 1.

Why might we be interested in knowing the genus of a graph? Such a question might arise in circuit design, for example, if we wanted to print electronic circuits on a circuit board to minimize crossings that could result in a short circuit. In this paper, we discuss how graphs embedded on surfaces of higher genus can be represented on the plane, how Kuratowski's Theorem can be extended to higher order surfaces, how Euler's Formula can be modified to account for the genus of the graph, and how we can determine the genus of a graph.

1.2 Surfaces of Higher Order

Since graphs drawn on three-dimensional surfaces are hard to work with, we would like to determine ways to draw graphs on 2-dimensional surfaces. If we consider a torus, we can create a two-dimensional representation of the surface by slicing the handle of the torus to create a cylinder and then slicing the cylinder lengthwise to create a rectangle (see Figure 1.3).

This rectangle has the property that if an edge stretches out to a side it continues back from the other side of the rectangle. Figure 1.4 shows an embedding of K_5 on this rectangle with no crossing edges. Many graphs can be drawn in S_1 with a very high level of symmetry. Figure 1.6 demonstrates two planar embeddings constructed in S_1: $K_{4,4}$ and K_7.

The surfaces that can be created by introducing handles to a sphere are all **orientable**. Intu-

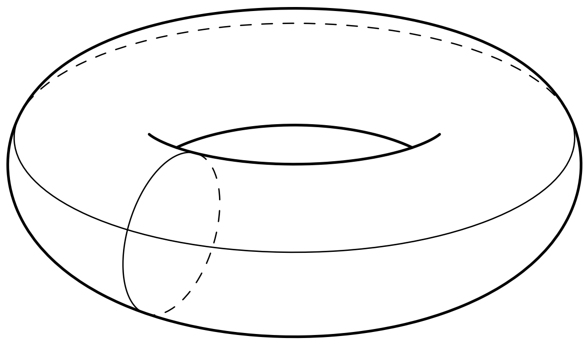


Figure 1.3: A torus representing the cuts required to transform the surface into a rectangle. The vertical cut produces a cylinder, and then with the horizontal cut the surface becomes a rectangle.

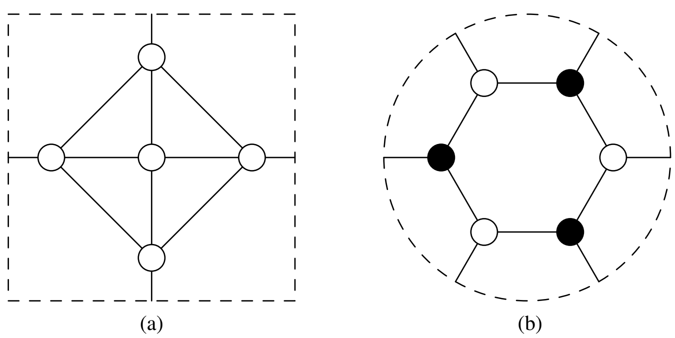
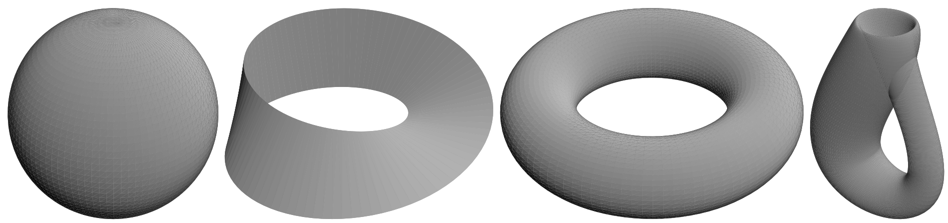


Figure 1.4: (a) A representation of K_5 embedded on the surface S_1, and (b) a representation of $K_{3,3}$ embedded on the projective plane.

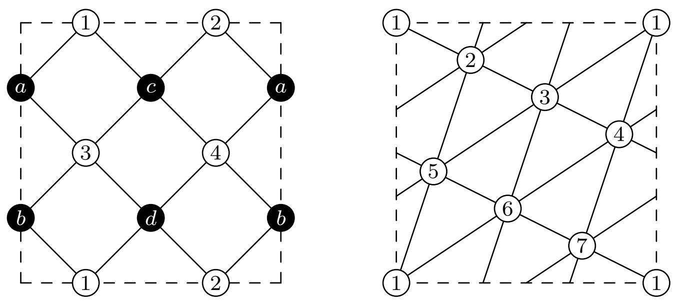
itively, a surface is said to be orientable if it has two distinct “sides.”[1] There are many interesting properties of **non-orientable surfaces** such as the Möbius strip N_1, and the Klein Bottle N_2, (see Figure 1.5), but the remainder of this paper will only address the orientable surfaces $S_0, S_1, \ldots$.[2] The graphs K_5 and $K_{3,3}$ are called **forbidden minors** in S_0. Formally, a forbidden minor in S_k is



(a) (b) (c) (d)

Figure 1.5: Representations of some common surfaces in 3-space: (a) The sphere, S_0; (b) The Möbius strip, N_1; (c) the torus, S_1; and (d) the Klein Bottle, N_2.

a graph G not embeddable in S_k with the property that, if we were to remove any edge from G, we would be left with a graph that is embeddable in S_k. By Kuratowski's Theorem, K_5 and $K_{3,3}$ are the unique forbidden minors in S_0.[3]



(a) (b)

Figure 1.6: (a) A planar embedding of $K_{4,4}$ in S_1, and (b) a planar embedding of K_7 in S_1.

1.3 Euler's Formula Extended

Euler's Formula states that for a planar embedding of a graph with V vertices, E edges, and F faces (or regions), we have the relation

$$V - E + F = 2.$$

[1]For example, the Möbius strip is not an orientable surface as it only has one “side.”

[2]However, we will mention one more noteworthy, non-orientable representation: the **projective plane** is a non-orientable surface that allows for a fairly straightforward embedding of the graph $K_{3,3}$ (see Figure 1.4b).

[3]It is a good exercise to try to find all forbidden minors in the space S_1. The reader may find it surprising to learn that there are in fact over 800 forbidden minors in S_1 alone [We], one example being the graph $2K_4 + K_1$, which is simply two copies of the complete graph on four vertices with an extra vertex adjacent to all the other vertices.

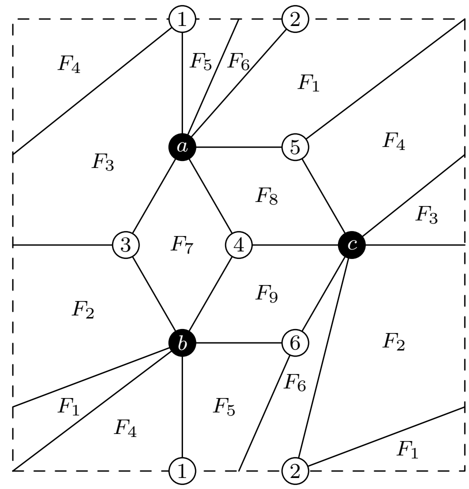


Figure 1.7: A planar embedding of $K_{3,6}$ in S_1. Note that for this embedding, $V - E + F = 0$.

If we examine a planar embedding of the graph $K_{3,6}$ on the surface S_1 (see Figure 1.7), we find that the genus-1 analogue of Euler's Formula is $V - E + F = 0$. This motivates the derivation of one of the most significant theorems in topological graph theory, an extension of Euler's Formula for higher-order surfaces:

$$V - E + F = 2 - 2g,$$

where g is the genus of the surface the graph is embedded upon, and the quantity $2 - 2g$ is defined to be the **Euler characteristic** $\chi(G)$ of the graph G.[4] Before proving this formula, we introduce some basic terminology; cf. [Wh].

Definition 2. A **pseudograph** is a graph with loops and multiple edges allowed.

Definition 3. A region of an embedding of a graph G in a surface M is said to be a **2-cell** if it is homeomorphic to the open disk. If every region of an embedding is a 2-cell, the embedding is said to be a **2-cell embedding**.

Here, when we say that a region is "homeomorphic to the open disk," we mean that it is topologically equivalent to a flat disk. If we were to shrink the the face of such a region to a point, we would find that it does not contain any irregularities such as handles or holes.

Theorem 4 (Euler's Formula)**.** *Let G be a connected pseudograph, with a 2-cell embedding in S_g, with the usual parameters V, E, and F. Then*

$$V - E + F = 2 - 2g. \tag{1.1}$$

Proof. We argue by induction on g. For the base case, we suppose $g = 0$. This reduces to the formula for planar graphs in S_0, $V - E + F = 2$, which we can prove by induction on the number of edges. If there are no edges, then G is an isolated vertex, and therefore $V - E + F = 1 - 0 + 1 = 2$. Otherwise, choose any edge e. If e is a loop, then remove it and E and F decrease by 1. If e connects two different vertices, contract e to a point and V and E each decrease by 1. In either case, the result follows by induction.

[4] Note that the Euler characteristic $\chi(G)$ should not be confused with the **chromatic number**, the least number of colors needed to color the vertices of a graph so that no adjacent vertices are of the same color.

Having shown the base case, we now assume the theorem holds true for graphs of genus $g-1$. We wish to show that a connected pseudograph G with a 2-cell embedding in S_g satisfies the formula. Let G be the graph of interest, with parameters V, E, and F.

Since the embedding is a 2-cell, this means that each face must have the property that it can be shrunk down to a point. Therefore if a face were to contain a handle, it would not be a proper 2-cell embedding. This implies that every handle must have at least one edge through it (see Figure 1.8).

Select one handle, and draw two closed curves C_1 and C_2 around the handle (by "around", we mean that if the surface were a coffee mug, you would draw out a curve by wrapping your hand around the handle) such that edges that intersect C_1 intersect C_2, and vice versa; this is always possible.

Suppose edges $e_1, e_2, \ldots, e_n$ run over the handle. Let x_{ij} be the point where curve C_i meets edge e_j, where $1 \leq i \leq 2$ and $1 \leq j \leq n$. Consider the points x_{ij} to be the vertices of a new pseudograph, whose edges consist of the appropriate subdivisions of the original edges, as well as the edges formed along the curves. Call this new graph G'; the graph G' is an extension of the graph G. It also includes the old edges and vertices outside the handle under consideration. We now have the parameters:

$$\begin{aligned} V' &= V + 2n, \\ E' &= E + 4n, \\ F' &= F + 2n. \end{aligned}$$

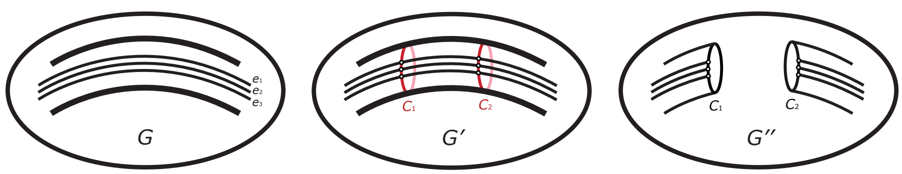


Figure 1.8: The handle under consideration shown in the graphs G, G', and G''.

Now, remove the portion of the handle between C_1 and C_2 and "fill in" the two resulting holes with two disks (the disks can be thought of as "caps"). Note that the edges that formed the curve are still present, and the caps they enclose account for two new faces. The result is the 2-cell embedding shown in Figure 1.8 of a connected pseudograph G'' in S_{g-1}, with the following parameters:

$$\begin{aligned} V'' &= V' = V + 2n, \\ E'' &= E' - n = E + 3n, \\ F'' &= F' - n + 2 = F + n + 2. \end{aligned}$$

By the inductive hypothesis, we now have:

$$\begin{aligned} 2 - 2(g-1) &= V'' - E'' + F'' \\ &= (V + 2n) - (E + 3n) + (F + n + 2) \\ &= V - E + F + (2n - 3n + n) + 2 \\ &= V - E + F + 2, \end{aligned}$$

the desired result. □

1.4 Determining the Maximum Genus of a Graph

Euler's Formula is a useful technique for finding upper and lower bounds for the genus of a graph. An important value used to develop many of the upper bounds for graphs is the **maximum genus**

$\gamma_M(G)$ of a graph.

Definition 5. The **maximum genus** $\gamma_M(G)$ of a connected graph G is the maximum genus among the genera of all surfaces in which G has a 2-cell embedding.

This maximum exists because for a graph to have a proper 2-cell embedding on a surface S_k, it must have at least one edge crossing each of the k handles. This leads to the (very loose) bound of $\gamma_M(G) \leq e+1$, where e is the number of edges in G. We should also consider the following theorem of Duke [Du] which gives a deeper understanding of a graph's ability to be embedded in the surface S_k:

Theorem 6 (Duke). *A connected graph G has a 2-cell embedding in S_k if and only if $\gamma(G) \leq k \leq \gamma_M(G)$.*

Theorem 6 tells us that if we can embed a graph into both a surface of genus n and into a surface of genus $m > n$, then we can embed the same graph onto any surface of genus g, where $n \leq g \leq m$. In developing upper bounds for the maximum genus, for the sake of algebra, it is helpful to introduce the following construction:

Definition 7. The **Betti number** $\beta(G)$ of a graph G having v vertices, e edges, and m components, is given by: $\beta(G) = e - v + m$.

Hence, $\beta(G) = e - v + 1$ for any connected graph G. We can now derive an upper bound for the genus of a connected graph using Euler's Formula: Let G be connected, with a 2-cell embedding in S_k. Then $f \geq 1$, and:

$$\begin{aligned} k &= 1 + \frac{e-v-f}{2} \\ &= \frac{e-v-(f-2)}{2} \\ &\leq \frac{e-v+1}{2} \\ &= \frac{\beta(G)}{2}. \end{aligned}$$

We then have the bound,

$$\gamma_M(G) \leq \left\lfloor \frac{\beta(G)}{2} \right\rfloor. \tag{1.2}$$

Graphs for which $\gamma_M(G) = \left\lfloor \frac{\beta(G)}{2} \right\rfloor$ are said to be **upper embeddable**. It has been shown that all complete n-partite graphs (graphs consisting of n completely interconnected, independent sets) are upper embeddable [KRW]. Using bound (1.2) for $\gamma_M(G)$, we can derive upper bounds for the maximum genera of the complete graphs K_n, complete bipartite graphs $K_{m,n}$, and the n-cube Q_n $(n \geq 2)$.[5] In particular,

$$\gamma_M(K_n) \leq \left\lfloor \frac{(n-1)(n-2)}{4} \right\rfloor, \tag{1.3}$$

$$\gamma_M(K_{m,n}) \leq \left\lfloor \frac{(n-1)(m-1)}{2} \right\rfloor, \tag{1.4}$$

$$\gamma_M(Q_n) \leq (n-2)2^{n-2}. \tag{1.5}$$

The above formulas can be proven to be equalities for the maximum genera of complete graphs [NSW], complete bipartite graphs [R1], and the n-cube [Za].

[5] The **n-cube** is the simple graph whose vertices are the k-tuples with entries in $\{0,1\}$ and whose edges are the pairs of k-tuples that differ in exactly one position. For example, Q_2 has the structure of a square, and Q_3 is the cube.

1.5 Lower Bounds for the Genus of a Graph

We now investigate a technique for computing lower bounds for the genera of some simple graphs. Recall that the **degree** of a vertex is the number of adjacent vertices and the **length** of a region is the length of the closed path that bounds the region. Let v_i be the number of vertices of degree i, and let f_i be the number of regions of length i. If we focus only on 2-cell embeddings of graphs with minimum degree 3 ($\delta(G) \geq 3$), also called **polyhedral** graphs, it follows that $v = \sum_{i\geq 3} v_i$ and $f = \sum_{i\geq 3} f_i$. By the well-known **degree-sum formula**, the sum of the degrees of all the vertices is equal to twice the number of edges in a graph. We then have:

$$2e = \sum_{i\geq 3} i \cdot v_i. \tag{1.6}$$

Also, since each edge separates two regions or belongs twice to a single region, summing the sides of each face double-counts the edges, whereby we have:

$$2e = \sum_{i\geq 3} i \cdot f_i. \tag{1.7}$$

The above results hold for all polyhedral graphs, which include the complete graphs on n vertices for $n \geq 3$, the complete bipartite graphs $K_{m,n}$ for $m, n \geq 3$, and the n-cubes Q_n for $n \geq 3$. Thus, for all polyhedral graphs, $2e = \sum_{i\geq 3} i \cdot f_i \geq \sum_{i\geq 3} 3 \cdot f_i = 3f$, and therefore $f \leq \frac{2}{3}e$. Also, for all triangle-free polyhedral graphs, $2e = \sum_{i\geq 4} i \cdot f_i \geq \sum_{i\geq 4} 4 \cdot f_i = 4f$, whence $f \leq \frac{1}{2}e$.

Using these two inequalities in conjunction with Euler's Formula, we can obtain lower bounds for all polyhedral graphs and triangle-free polyhedral graphs. First consider the former, where we have $f \leq \frac{2}{3}e$. Using Euler's Formula and solving for g, we find:

$$\begin{aligned} g &= 1 - \frac{v}{2} + \frac{e}{2} - \frac{f}{2} \\ &\geq 1 - \frac{v}{2} + \frac{e}{2} - \frac{1}{2}\left(\frac{2}{3}e\right) \\ &= 1 - \frac{v}{2} + e\left(\frac{1}{2} - \frac{1}{3}\right) \\ &= 1 - \frac{v}{2} + \frac{e}{6}. \end{aligned}$$

Thus, we have the bound

$$\gamma(G) \geq \left\lceil 1 - \frac{v}{2} + \frac{e}{6} \right\rceil. \tag{1.8}$$

We can also develop the bound for triangle-free graphs in the same way to obtain:

$$\gamma(G) \geq \left\lceil 1 - \frac{v}{2} + \frac{e}{4} \right\rceil. \tag{1.9}$$

This gives that

$$\gamma(K_n) \geq \left\lceil 1 - \frac{n}{2} + \frac{n(n-1)}{12} \right\rceil \tag{1.10}$$

$$= \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil. \tag{1.11}$$

In fact [Ha, p. 118-119], we have equality, i.e.,

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \qquad n \geq 3. \tag{1.12}$$

An interesting result to note is that by the inequality (1.3), which is in fact an equality (cf. [NSW]) for large n, $\gamma_M(K_n) \to 3\gamma(K_n)$; this gives a range for the possible surfaces on which a 2-cell embedding of K_n can exist.

Similarly, we have the formula

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \qquad m,n \geq 2. \tag{1.13}$$

We can determine a lower bound for the genus of the n-cube Q_n in the same fashion, by using the triangle-free inequality (1.9):

$$\gamma(Q_n) \geq 1 + (n-4)2^{n-3}, \qquad n \geq 2. \tag{1.14}$$

It turns out that this is an equality. This was proven to be the genus of the n-cube Q_n in [R2]. The proof involves induction and a more complex technique called “surgery” on the graph [GT] which unfortunately is beyond the scope of this paper.

1.6 Conclusion

We have given several non-trivial bounds on the genera of certain families of graphs, as well as explicit formulas for a few highly symmetric families of graphs (namely the complete and complete bipartite graphs). Determining the genera of more complicated families of graphs usually involves calculating lower bounds using Euler’s Formula and then deriving (often using induction) a general construction of an embedding using techniques such as surgery.

One topic we have omitted entirely is the question of finding algorithms to determine the genus of a graph. However, this is a very interesting subject; it has been proven that there exists a linear time algorithm which finds an embedding of G in a surface S, or if this is impossible, finds a subgraph $K \subseteq G$ which is a subdivision of some forbidden minor of S (see [M1]). A general formula for the genus of an arbitrary graph is not known, but using the techniques discussed in this paper, many equalities can be constructed for n-partite graphs.

References

[BL] C. Paul Bonnington and Charles H. C. Little: *The Foundations of Topological Graph Theory*. New York: Springer-Verlag, 1995.

[Du] Richard A. Duke: The genus, regional number and Betti number of a graph, *Canad. J. Math* **18** (1966), 817–822.

[GT] Jonathan L. Gross, Thomas W. Tucker: *Topological Graph Theory*. New York: John Wiley & Sons, 1987.

[Ha] Frank Harary: *Graph Theory*. Redding, Massachusetts: Addison Wesley Publishing Company, 1969.

[KRW] Hudson V. Kronk, Richard D. Ringeisen, and Arthur T. White: On 2-cell imbeddings of complete n-partite graphs, *Colloq. Math.* **36** (1976), 131–140.

[Ku] Kazimierz Kuratowski: Sur le problème des courbes gauches en topologie (in French), *Fund. Math.* **15** (1930), 271–283.

[M1] Bojan Mohar: A linear time algorithm for embedding graphs in an arbitrary surface, *SIAM J. Discrete Math.* **13** (1999), 6–26.

[M2] Bojan Mohar and Carsten Thomassen: *Graphs on Surfaces*. Baltimore: Johns Hopkins University Press, 2001.

[NSW] E. A. Nordhaus, B. M. Stewart, and Arthur T. White: On the maximum genus of a graph, *J. Combinatorial Theory* B **11** (1971), 258–167.

[R1] Richard D. Ringeisen: Determining all compact orientable 2-manifolds upon which $K_{m,n}$ has 2-cell imbeddings, *J. Combinatorial Theory* B **12** (1972), 101–104.

[R2] Gerhard Ringel: Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter (in German), *Abh. Math. Sem. Univ. Hamburg* **20** (1955), 10–19.

[We] Douglas B. West: *Introduction to Graph Theory*, 2nd Edition. Upper Saddle River: Prentice-Hall, 2001.

[Wh] Arthur T. White: *Graphs of Groups on Surfaces: Interactions and Models*. New York: Elsevier Science, 2001.

[Za] J. Zaks: The maximum genus of cartesian products of graphs, *Canad. J. Math.* **26** (1974), 1025–1036.