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Solving Large Classes of Nonlinear Systems of PDEs by the Method of Order Completion

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"... provided also if need be that the notion of a solution shall be suitably extended."

—Hilbert's 20th Problem

6.1 Preliminaries

One of the sharpest divides in the history of technological and mathematical development occurred with Newton's development of calculus. Prior to this development, we were not able to understand, let alone model rigorously, the motion of even a single massive particle unless it was moving along a straight line or a circle and was doing so with constant velocity. Even Galileo's discoveries about gravitation and Kepler's laws of planetary motion were merely *empirical*. In other words, these so-called laws were not based on any scientific principles or corresponding rigorous theories and were instead just formulae fitted to observed data.

Newton's three laws of motion and the methods of calculus he invented enabled the practice of science as we know it today. In particular, Newton's methods allowed the fundamental laws of nature to be formulated as **differential equations**. In fact, much of modern science would be impossible to formulate, let alone apply technologically, without the use of ordinary and partial differential equations, denoted respectively by ODEs and PDEs.

A simple example serves to illustrate the immense leap brought about by Newton's calculus. **Newton's Second Law** states, in modern terms, that the motion of a massive particle along a straight line satisfies the property "mass times acceleration is equal to force," $ma = F$. In terms of the position $x(t)$ of a particle, acceleration is the second derivative: $a(t) = \ddot{x}(t)$. Thus the Second Law takes the form of the second-order ordinary differential equation $m\ddot{x}(t) = F(t)$ in the

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position $x(t)$. A significant implication of this law is that one requires precisely *two* independent initial conditions to determine a unique solution in a particular physical situation. One may, for example, give the initial position $x(0)$ and the initial velocity $\dot{x}(0)$.

While this may seem obvious, Newton's innovation was in fact quite revolutionary. For approximately two millennia prior to Newton, the prevailing view of such motion, stated by Aristotle, held velocity, rather than acceleration, to be proportional to the force applied. Thus, according to Aristotle, the *first*-order differential equation $\dot{x}(t) = cF(t)$, where $c > 0$ is some constant, would describe motion; he engaged in no experimentation to test this notion. Using Newton's calculus, the trouble with such a first-order equation is so obvious that one need not conduct any experimentation to perceive it. Indeed, as the equation is first-order in the position $x(t)$, it only allows *one single* initial condition for the unique determination of its solution. But this *contradicts* the empirically known fact that one can give two objects with the same initial position two different initial velocities, resulting in two different trajectories.

Considerable mathematical effort has been expended in solving such equations, especially PDEs. The modern era of PDE theory started in the early 20th century, when methods of functional analysis were introduced. This trend became very strong starting in the 1930s, when a large variety of Sobolev spaces proved to be particularly convenient for finding solutions to PDEs. Finally, in the late 1940s, with the introduction of Schwartz distributions, the functional analytic methods became ubiquitous.

Sobolev spaces are complete normed spaces, that is, **Banach spaces**; they are more sophisticated than the **Lebesgue spaces** L^p, with $1 \leq p \leq \infty$, since their norms involve not only the generalized functions which are their elements but also various derivatives of these generalized functions. A motivation for such sophisticated norms is that, under the usual norms of the L^p spaces, the derivative is not a bounded operator and thus is not continuous. Under certain conditions, the generalized functions in Sobolev spaces turn out to be the usual smooth functions.

As for the more general spaces $\mathcal{D}'$ or $\mathcal{S}'$ of **Schwartz distributions**, these are no longer normed spaces, so their topologies are far more complicated. Indeed, those topologies are locally convex, thus considerably more general than those of normed spaces. These more general spaces of distributions prove to have advantages beyond those of Sobolev spaces when solving certain classes of PDEs.

Let us now be more explicit about what it means to solve a differential equation, from both a mathematical and physical perspective. We take two simple yet nontrivial examples for illustrative purposes.

First, let us consider the motion of a particle under gravitation. We let the variable x denote position and denote by x_0 the position of the particle at time $t = 0$. Then, as gravitational force is constant, Newton's Second Law gives the second-order ordinary differential equation $\ddot{x}(t) = 1$ after an appropriate normalization of units. A general solution of this equation exists for all $t \in \mathbb{R}$, given by $x(t) = t^2/2 + v_0 t + x_0$, where v_0 is the initial velocity of the particle. Now two facts are important to note here: First, such a general solution exists for all $t \in \mathbb{R}$. Second, the general solution describes all possible free falls, rather than merely the free fall of one particular particle. To find this description of the particle's motion, we specify its initial position x_0 together with its initial velocity v_0. In general, we are interested in **existence** and **uniqueness** of solutions, the latter under specific additional conditions which are required by, for example, physical reality.

As a second example, let us consider one of the most basic PDEs of fluid dynamics, namely the **nonlinear shock-wave equation** $U_t(t, x) + U(t, x)U_x(t, x) = 0$, where $t \geq 0, x \in \mathbb{R}$. Under certain conditions, this equation can be seen as describing the motion of a fluid within an infinitely long tube parallel to the x-axis. Here, $U(t, x)$ represents the velocity at time t of a particle of fluid which is at the point of coordinate x. It is well known that the general solution of this equation exists; the issue is how to determine a *unique* solution which corresponds to a specific physical situation.

The mathematical answer, and one which makes physical sense, is that one should give an initial condition which describes the velocity of the fluid at time $t = 0$ and along the whole x-axis. Namely, one has to give $U(0, x) = u(x)$, for $x \in \mathbb{R}$; in this case, the initial condition is a function defined on all of $\mathbb{R}$.

There is also a *third* problem, especially concerning the solution of PDEs, namely, determining the **regularity** of solutions. From this point of view, the nonlinear shock-wave equation can already give a good example, even if it is one of the simplest nontrivial nonlinear equations of major physical interest. Namely, we note that this equation is of *first*-order, since only first-order partial derivatives of the unknown function U appear. Therefore, we would expect that any well-behaved solution is given by a function U of t and x such that both the partial derivatives U_t and U_x which appear in that equation exist. Indeed, if we are given a function U of t and x which is not differentiable with respect to both variables, then we simply cannot verify in a straightforward manner whether U is a solution of that equation. So, by a **classical solution** to a PDE, we mean a function for which all partial derivatives which appear in the PDE exist; this condition is therefore the natural definition of regularity.

Here, however, things get rather complicated. Indeed, even in the case of the above nonlinear shock-wave equation, many physically relevant solutions are not at all classical. Such solutions are called **shock waves**, and their existence and physical relevance is precisely the reason for the name of the equation. In fact, such shock wave solutions U not only lack the partial derivatives U_t and U_x, but even fail to be continuous. Nonetheless, such solutions are physically realistic; for example, they can model the effects of a sonic boom.

It follows that even though we would like solutions to be regular in the classical sense, important practical considerations oblige us to deal with solution which are less regular than the classical ones. Such non-regular solutions are then called **generalized solutions**. Of course, developments show that such generalized solutions do nevertheless satisfy the respective PDEs in certain suitable senses. Consequently, the problem of regularity of solutions of PDEs means, in practice, to find solutions which are in some sense "generalized as little as possible" and thus are as near to classical solutions as possible. The case of the shock waves shows that this is not always a trivial issue.

The history of solving ordinary and partial differential equations is impressively rich and complex. Its complexity should in no way be surprising, since most of the fundamental laws of nature which such equations model are not as simple as that of a free-falling particle. The history is also remarkable, given its sometime paradoxical ways of progressing.

For instance, in spite of the fact that solving PDEs is significantly harder than solving ODEs, the first general existence, uniqueness and regularity result for solutions was that of Cauchy-Kovalevskaya for arbitrary nonlinear systems of analytic PDEs, obtained in the early 1870s. Furthermore, there are two instructive facts with respect to this result. First, the "hardest" mathematics used in the proof of that theorem is the summation of a convergent geometric series. Thus in particular, its proof used no topology, let alone functional analysis of any kind. Second, the subsequent century-long development of topology and functional analysis was not able to improve even slightly upon the original result of the Cauchy-Kovalevskaya theorem when one considers this theorem in its own terms of nonlinear generality or upon the strength of its existence, uniqueness and regularity results. In this regard, the first time an extension in of the Cauchy-Kovalevskaya theorem (its own terms) was obtained was in [Ro7].[1] Once again (and surprisingly), functional analytic methods were not used.

As it happens, it took approximately two decades following the Cauchy-Kovalevskaya theorem before a correspondingly general existence and uniqueness result for ODEs was obtained by Picard and Lindelöf, who used a sophisticated fixed point argument, typical of methods in functional analysis. The relative strength of the Picard-Lindelöf result is that it is valid not only for analytic ODEs, but also for those ODEs which are far less smooth, for instance, ODEs that are continuous with some mild Lipschitz-type conditions.

As far as ODEs are concerned, methods of solving such equations are well-established, and the main remaining concerns are of a numerical nature related to improvements in the approximation of such solutions. With respect to the solution of PDEs, since the introduction of Sobolev spaces, and in general, of the Schwartz distributions, functional analytic methods have attained a near-monopoly, with hardly any other significant methods developed until the late 1970s. Furthermore, it became common to claim that it is simply not possible mathematically to develop a general

[1] See also [Ro8], [Ro12], where a global version of that theorem was presented.

existence, uniqueness and regularity theory for solving PDEs.

Instead, it is claimed, one must focus on specific types of such equations, each with its own highly specific solution method. Thus, the claim is that present day mathematics is in fact incapable of developing any relevant **type-independent** PDE theory with respect to the existence, uniqueness and regularity of solutions. Recent expressions of that strongly entrenched view can be seen in advanced textbooks of noted specialists in PDEs. For example Arnold's text [Ar], starts with the statement (italics added):

> In contrast to ordinary differential equations, there is *no unified theory* of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry...

Similarly, Evans' text [Ev], starts his Examples on page 3 with the somewhat more cautious statement (italics added):

> There is no general theory known concerning the solvability of all partial differential equations. Such a theory is *extremely unlikely* to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modelled by PDE. Instead, research focuses on various particular partial differential equations ...

The historical facts, however, show the relevance of general, type-independent results concerning PDEs. Indeed, in the context of arbitrary **analytic nonlinear** systems of PDEs, such a general, type-independent result was obtained on the existence, uniqueness, and analytic regularity of solutions back in the 1870s with the classical Cauchy-Kovalevskaya theorem. In the context of **linear constant coefficient** PDEs, a general type-independent existence result was already obtained in the early 1950s by Malgrange, and independently by Ehrenpreis, concerning the so-called elementary solutions of such equations, related to the well-known Green functions.

The severe limitations of the functional analytic methods in solving even linear PDEs came most unexpectedly and shockingly to the fore fifty years ago, with the celebrated 1957 Hans Lewy impossibility result [Le], concerning the nonexistence of solutions of PDEs. Indeed, Lewy showed that the rather simple linear first-order PDE in three independent variables and with first degree polynomial coefficients

$$(D_x + iD_y - 2(x+y)D_z)U(x,y,z) = f(x,y,z), \qquad (x,y,z) \in \mathbb{R}^3 \tag{6.1}$$

does not have distribution solutions in any neighborhood of any point in $\mathbb{R}^3$, for a large class of smooth right-hand terms f. In 1967, Shapiro gave a similar example of a smooth linear PDE which does not have solutions in Sato's **hyperfunctions**.

Recently, however, type-independent existence, uniqueness, and regularity results on solutions of large classes of nonlinear systems of PDEs, with possibly associated initial and/or boundary value problems have been introduced [OR]. The method of solution, a first in the literature, is based on the **order completion** (see Appendix) of suitable spaces of usual functions on the Euclidean domains of definition of the respective PDEs. As a general and hence type-independent regularity result, the solutions obtained can be assimilated with **Hausdorff continuous** functions (see Appendix) on the domains of the PDEs (see [An], [Ro1, Ro9, Ro11, Ro17], [Wa1, Wa2, Wa3, Wa4]).

Thus, one can do away with the use of various generalized functions, such as Schwartz distributions or elements of Sobolev spaces.[2] However, the latest methods of improved regularity being developed for the general order completion method indicate the possibility of gaining more insight into this problem [Wa1, Wa2, Wa3, Wa4]. An important fact to note is that the order completion solution method does not involve functional analysis; thus, it does not make use of various Sobolev or other spaces of distributions or generalized functions which usually provide solutions to PDEs.

[2] Among others, the usual 3-dimensional Navier-Stokes equations are included as a particular case of the nonlinear systems of PDEs which can be solved by the order completion method; the resulting solutions are Hausdorff continuous (a somewhat weak regularity condition).

Instead, the solutions obtained are no longer generalized functions and can be assimilated with Hausdorff continuous functions.

The power of the order completion method is shown in three facts. First, this method is the first in the literature to overcome the celebrated 1957 Hans Lewy impossibility . In fact, it overcomes Lewy's result in the case of very general nonlinear PDEs, far beyond the simple linear PDE in (6.1). Second, the order completion solution method allows a particularly convenient treatment of initial and/or boundary value problems associated with PDEs, which, as is well-known, is an advantage over functional analytic methods [OR, Chapter 8]. Third, and perhaps most importantly, the concept of order is more basic than that of algebraic structure. Indeed, the dichotomy between linear and nonlinear PDEs, which singles out the nonlinear ones as incomparably harder to solve, manifests itself on the algebraic level: more precisely, in terms of vector spaces. Therefore, that unfortunate dichotomy between linear and nonlinear PDEs is simply unavoidable with functional analytic methods. On the other hand, the order completion method does not distinguish between linear and nonlinear PDEs, solving both types of differential equation with equal ease (see [OR], [Ro1, Ro9, Ro11, Ro17], [Wa1, Wa2, Wa3, Wa4]).

The order completion method in [OR] brought a considerable improvement with respect to the extent that general type-independent existence, uniqueness and regularity results concerning solutions of large classes of nonlinear systems of PDEs can be obtained. Thus, it appears that the limitations claimed on PDE theory are in fact only limitations on the functional analytic methods used.

6.2 Main Ideas of the Order Completion Solution Method

The solution method is divided into two parts. The proof of the **existence** and **uniqueness** of solutions follows the method of order completion introduced and first developed in [OR]. The proof of the **regularity** of solutions is a consequence of recent results regarding the structure of the **Dedekind order completion** of spaces of continuous functions $\mathcal{C}(X)$, where X is a topological space with some weak conditions on it [An]. The respective regularity results have further been developed and improved in [Ro1, Ro9, Ro11, Ro17], [Wa1, Wa2, Wa3, Wa4].

For simplicity of presentation, we shall consider single nonlinear PDEs.[3] Let us therefore consider nonlinear PDEs of the general form

$$F(x, U(x), \ldots, D_x^p U(x), \ldots) = f(x), \qquad x \in \Omega \subseteq \mathbb{R}^n \tag{6.2}$$

with $p \in \mathbb{N}^n$, $|p| \leq m$. Here, the domain Ω is an open, not necessarily bounded subset of $\mathbb{R}^n$, while the orders $m \in \mathbb{N}$ of the PDEs are fixed but otherwise arbitrary, and solutions are functions $U : \Omega \to \mathbb{R}$.

The unprecedented generality of these nonlinear PDEs comes, above all, from the class of functions F which define the left-hand terms, and which are only assumed to be **jointly continuous** in all of their arguments. The right hand terms f are also required to be continuous.[4]

Regardless of the above generality of the nonlinear systems of PDEs considered, one can find for them solutions U defined on the whole of the respective domains Ω. These solutions U have the **type-independent**, or **universal regularity**, property that they can be assimilated with Hausdorff continuous functions.

It follows in this way that, when solving systems of nonlinear PDEs of the generality of those in (6.2), one can dispense with the various customary spaces of distributions, hyperfunctions, generalized functions, Sobolev spaces, and so on. Instead, one can stay within the realms of "usual

[3] The extension to systems of such nonlinear PDEs and associated initial and/or boundary value problems can, rather surprisingly, be done easily, this being one of the major advantages of the order completion method (see [OR]).

[4] However, it turns out that in the most general case, both F and f can have certain discontinuities as well (see [OR]).

functions," that is, **interval-valued functions** (see Appendix).[5]

Let us now associate with each nonlinear PDE in (6.2) the corresponding nonlinear partial differential operator defined by its left hand side, namely

$$T(x,D)U(x) = F(x, U(x), \ldots, D_x^p U(x), \ldots), \qquad x \in \Omega. \tag{6.3}$$

The fact that $T(x, D)$ is an **operator** simply means that the nonlinear PDE in (6.2) can be written in the simple form

$$T(x,D)U(x) = f(x), \qquad x \in \Omega. \tag{6.4}$$

Two facts about the nonlinear PDEs in (6.2) and the corresponding nonlinear partial differential operators $T(x, D)$ in (6.3) are important and immediate:

- The operators $T(x, D)$ can *naturally* be seen as functions acting in the **classical context**, namely, between classical spaces of functions

$$T(x,D) : \mathcal{C}^m(\Omega) \to \mathcal{C}^0(\Omega). \tag{6.5}$$

  Unfortunately on the other hand:

- The mappings in this natural classical context (6.5) are typically not surjective even in the case of linear $T(x, D)$, and they are even less so in the general nonlinear case of (6.2), (6.4).

In other words, linear or nonlinear PDEs in (6.2) typically cannot be expected to have classical solutions $U \in \mathcal{C}^m(\Omega)$, for arbitrary continuous right-hand terms $f \in \mathcal{C}^0(\Omega)$, as illustrated by a variety of well-known examples, some of them rather simple ones (see [OR, Ch. 6]).

Furthermore, it can often happen that non-classical solutions do have a major applicative interest and thus have to be sought out beyond the confines of the classical framework in (6.5). One of the simplest such examples comes from the aforementioned shock wave solutions of the nonlinear shock wave equation. In fact, non-classical solutions can be critically important, even in the case of linear PDEs.[6] Thus we are led to the necessity of considering generalized solutions U to PDEs like those in (6.2), that is, solutions $U \notin \mathcal{C}^m(\Omega)$, which therefore are no longer classical. This means that the natural classical mappings (6.5) must in certain suitable ways be extended to **commutative diagrams**:

$$\begin{array}{ccc} \mathcal{C}^m(\Omega) & \xrightarrow{T(x,D)} & \mathcal{C}^0(\Omega) \\ \big\downarrow \subseteq & & \subseteq \big\downarrow \\ X & \xrightarrow{\widetilde{T}} & Y \end{array} \tag{6.6}$$

The generalized solutions are now found as

$$U \in X \setminus \mathcal{C}^m(\Omega), \tag{6.7}$$

instead of the classical solutions $U \in \mathcal{C}^m(\Omega)$, which may easily fail to exist. A further important point is that one expects to reestablish certain kinds of surjectivity properties typically missing in (6.5); for example,

$$\mathcal{C}^0(\Omega) \subseteq \widetilde{T}(X). \tag{6.8}$$

[5] Furthermore, when proving the existence and the mentioned type of regularity of such solutions, one can dispense with methods of functional analysis. However, functional analytic methods can possibly be used in order to obtain further regularity or other desirable properties of such solutions. Therefore, the order completion method does not aim to abolish functional analytic methods in solving PDEs, but rather to improve significantly on the well-known—yet so often disregarded—severe limitations of such methods.

[6] Such, for example, as those whose solutions are given by **Green functions**.

Here, it is important to note the following two facts. First, the extended spaces X and Y need not be minimal. Indeed, one is interested in solving not only one particular PDE, or one single system of PDEs. On the other hand, as the history of PDE theory has clearly shown, we cannot expect to find some sort of universally valid unique extensions X or Y. Moreover, such extensions may often depend on the PDEs solved, although different PDEs may still be solvable in the same extensions. Second, and following from the above, we should not always ask the surjectivity condition in its strongest possible form, $\widetilde{T}(X) = Y$. Instead, depending on the particulars of the situation, it may be sufficient to ask only that $\widetilde{T}(X)$ is a large enough subset of Y, such as that specified in (6.8) above.

Before going further, let us recall that extensions of mappings through commutative diagrams similar to (6.6) have been associated with solving equations—even if not explicitly—ever since ancient times (see [OR, chap. 12]). For example, it is well-known that for all $x \in \mathbb{R}$, $x^2 \neq -1$. That is, $x^2 + 1 = 0$ has no solution in $\mathbb{R}$. However, as we all know, that equation does have a solution in $\mathbb{C}$. This fact can be formulated in the following extension of a mapping through a commutative diagram. Namely, let us define the mapping $T : \mathbb{R} \ni x \mapsto T(x) = x^2 \in \mathbb{R}$. Then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{T} & \mathbb{R} \\ \downarrow \subseteq & & \subseteq \downarrow \\ \mathbb{C} & \xrightarrow{\widetilde{T}} & \mathbb{C} \end{array} . \tag{6.9}$$

Here, of course, T is *not* surjective, since $-1 \in \mathbb{R} \setminus T(\mathbb{R})$. On the other hand, $\widetilde{T} : \mathbb{C} \ni x \mapsto \widetilde{T}(x) = x^2 \in \mathbb{R}$ has the property that $-1 \in \widetilde{T}(\mathbb{C})$.

6.3 Constructing the Order Completion

Since we solve PDEs through order completion, let us see how close we can come to satisfying the equality in (6.2), in the sense of order. For that purpose, it is useful to consider, for each $x \in \Omega$, the set of real numbers

$$\mathbb{R}_x = \{F(x, \xi_0, \ldots, \xi_p, \ldots) \mid \xi_p \in \mathbb{R}, \text{ for } p \in \mathbb{N}^n, |p| \leq m\}. \tag{6.10}$$

Clearly, for fixed $x \in \Omega$, $\mathbb{R}_x$ is the range in $\mathbb{R}$ of $F(x, \ldots)$, and since F is jointly continuous in all its arguments, it follows that $\mathbb{R}_x$ is a nonempty interval which is bounded, half-bounded, or is the whole of $\mathbb{R}$. This latter case, which can happen often with nonlinear PDEs in (6.2), will be easier to deal with, as we will see in (6.12) below.

Clearly, in the case of non-degenerate linear PDEs in (6.2), the latter case is ubiquitous. Now given $x \in \Omega$, it is obvious that a necessary condition for the existence of a classical smooth solution $U \in \mathcal{C}^m$ of (6.2) in a neighborhood of x is

$$f(x) \in \mathbb{R}_x. \tag{6.11}$$

Consequently, for the time being, we shall make the assumption that the right hand term functions f in the nonlinear PDEs in (6.2) satisfy the somewhat stronger version of condition (6.11) given by

$$f(x) \in \text{interior}(\mathbb{R}_x), \qquad \text{for } x \in \Omega. \tag{6.12}$$

Clearly, whenever we have

$$\mathbb{R}_x = \mathbb{R}, \qquad \text{for } x \in \Omega, \tag{6.13}$$

then (6.12) is satisfied. And as mentioned, this is the case with all nontrivial linear PDEs, as well as with most of the nonlinear PDEs of practical interest.

We now formulate the basic and rather simple **local approximation** result on how nearly we can satisfy the equality in (6.2) and (6.4). A remarkable fact is that the proof of this local approximation result, as well as of its global version in Proposition 2 in the sequel, is surprisingly elementary.

Proposition 1 ([OR], Lemma 2.2)**.** *Given* $f \in \mathcal{C}^0(\Omega)$*, then for all* $x_0 \in \Omega \subset \mathbb{R}^n$, $\epsilon > 0$, *there exists* $\delta > 0$ *and a polynomial* P *in* n *variables with real coefficients such that in a* δ*-ball around* x_0*, we have*

$$f(x) - \epsilon \leq T(x, D)P(x) \leq f(x). \tag{6.14}$$

In view of the several successive quantifiers in the above approximation result, let us briefly elucidate it in a somewhat less formal manner. Our first aim is, of course, to prove the existence of solutions U of the nonlinear PDE in (6.4). The order completion method obtains such existence results in two steps. First, it shows that the nonlinear PDE in (6.4) can be satisfied *approximately* as nearly as we want. Second, it shows that—in their totality as a set—such approximate solutions do in fact define an exact solution, provided that we build a convenient order completion of both the domain and range of the nonlinear partial differential operator $T(x, D)$ in (6.3) and do so as in the commutative diagram (6.6). This is similar to the order completion of the rationals used to obtain the reals and in fact uses a method analogous to Dedekind cuts. As it happens, however, with the nonlinear PDE in (6.4), we are not looking for one single number, but for a whole function $U : \Omega \to \mathbb{R}$. Furthermore, it is much easier first to approximate the solution of that nonlinear PDE in (6.4) only *locally*, that is, in a suitable neighborhood of any given point $x_0 \in \Omega$. This approximation is precisely what the above proposition accomplishes. Namely, for every given $x_0 \in \Omega$ and $\epsilon > 0$, it delivers such a simple (in fact, polynomial) function P, together with a neighborhood of x_0 described by a corresponding $\delta > 0$, with the **two-sided approximation property**

$$f(x) - \epsilon \leq T(x, D)P(x) \leq f(x), \qquad x \in \Omega, \|x - x_0\| \leq \delta.$$

Let us briefly give a proof of Proposition 1:

Proof of Proposition 1. Let any $x_0 \in \Omega$ be given. Then for suitable $\epsilon > 0$, (6.12) yields

$$\xi_p \in \mathbb{R}, \qquad \text{for } p \in \mathbb{N}^n, |p| \leq m$$

such that

$$F(x_0, \xi_0, \ldots, \xi_p, \ldots) = f(x_0) - \frac{\epsilon}{2}$$

Therefore, there exists a polynomial P in the variable $x \in \mathbb{R}^n$, such that

$$D^p P(x_0) = \xi_p, \qquad \text{for } p \in \mathbb{N}^n, |p| \leq m,$$

which means that

$$T(x_0, D)P(x_0) - f(x_0) = -\frac{\epsilon}{2}.$$

However, both F and f are assumed to be continuous, so the function $\Omega \ni x \mapsto T(x, D)P(x) - f(x) \in \mathbb{R}$ is continuous as well. Therefore (6.14) follows immediately. □

And now, the **global approximation** version of the inequality property in (6.14) is given by

Proposition 2 ([OR], Prop. 2.2)**.** *Suppose* $f \in \mathcal{C}^0(\Omega)$*. Then for all* $\epsilon > 0$*, there exists* $\Gamma_\epsilon \subset \Omega$ *closed and nowhere dense and* $U_\epsilon \in \mathcal{C}^m(\Omega \setminus \Gamma_\epsilon)$ *such that*

$$f - \epsilon \leq T(x, D)U_\epsilon \leq f \tag{6.15}$$

on $\Omega \setminus \Gamma_\epsilon$.

Remark. It is easy to see that the inequalities in (6.14) and (6.15) can be replaced with

$$f(x) \leq T(x, D)P(x) \leq f(x) + \epsilon, \tag{6.16}$$

$$f \leq T(x, D)U_\epsilon \leq f + \epsilon, \tag{6.17}$$

as the proofs of (6.16) and (6.17) follow after the corresponding obvious minor changes in the proofs of the above two propositions.

We now proceed to the order completion, based on MacNeille's construction, using **Dedekind cuts** (see [OR, Ma, Lu]); such cuts require the above sharp inequalities. Let us briefly recall here Dedekind's original construction of $\mathbb{R}$ from $\mathbb{Q}$. While this construction is simpler than that of MacNeille, as $\mathbb{Q}$ is totally ordered, it is largely analogous.

Dedekind calls a **cut** in $\mathbb{Q}$ any partition of $\mathbb{Q}$ into two subsets A and B which satisfy $x < y$ for all $x \in A, y \in B$. For instance, the cut which defines $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ is given by $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$ and $B = \{y \in \mathbb{Q} \mid y^2 > 2\}$. Thus, if we want effectively to construct A, for example, then one way to obtain it is by the union $A = \bigcup_{\epsilon > 0} \{x \in \mathbb{Q} \mid 2 - \epsilon \leq x^2 \leq 2\}$ which also gives an approximation process (from below) for $\sqrt{2}$. The approximation results given here model this process, albeit with polynomials in place of rationals.

Note that in Proposition 2, as well as in its version corresponding to the above inequality (6.17), we can have in addition the property

$$\text{mes}(\Gamma_\epsilon) = 0 \tag{6.18}$$

where mes denotes the usual Lebesgue measure.[7]

As seen from the proof of Proposition 2 (see [OR, pp. 18-20]), the functions U_ϵ can in fact be chosen as piecewise polynomials in $x \in \mathbb{R}^n$.

The considerable power of the order completion method in solving very general classes of nonlinear systems of PDEs comes from the fact that in the above order approximation results (6.15) and (6.17), one does not need more than the continuity of the functions F and f which define the nonlinear PDEs (6.2). Due to the inevitable presence of the closed, nowhere dense subsets of singularities Γ_ϵ, one can in fact allow even certain *discontinuities* in these functions F and f (see [OR]).

And now, the construction of commutative diagrams (6.6) follows easily, [OR], [Ro1, Ro9, Ro11, Ro17]. Indeed, the order approximations in (6.15) or (6.17) lead to the construction of the spaces X and Y as the **Dedekind order completion** (see Appendix) of spaces of piece-wise smooth functions corresponding in a natural manner to $\mathcal{C}^m(\Omega)$ or $\mathcal{C}^0(\Omega)$. Then (this is nontrivial), the mappings $\widetilde{T}$ turn out to be **order isomorphic embeddings**.

6.4 General Existence Result

Once we reformulated the problem of solving PDEs in terms of the commutative diagrams (6.6), all the subsequent results concerning existence, uniqueness and regularity of solutions are obtained in terms of such diagrams.

One of the typical **main existence results** concerning the solutions of the nonlinear PDEs in (6.2) is presented in the following theorem (see [OR, pp. 38-64] for a proof):

Theorem 3. *In the commutative diagram* (6.6)*, we have*

$$\widetilde{T}(X) = Y. \tag{6.19}$$

That is, $\widetilde{T}$ *is surjective.*

[7] It should be noted that the presence of the closed, nowhere-dense singularity sets Γ_ϵ in the global inequalities (6.15) and (6.17) proves not to be a hindrance. In fact, the presence of such closed, nowhere dense singularity sets is rather deeply-rooted, as it is connected with the flabbiness of related sheaves of functions, or the global version of the classical Cauchy-Kovalevskaya theorem on analytic nonlinear PDEs (see [OR, chap. 7] and the literature cited there).

This means that, given any nonlinear PDEs in (6.2), for every right hand term $f \in Y$, there exists a solution $U \in X$, satisfying the relation $\widetilde{T}(U) = f$.

However, as mentioned following (6.8) and (6.9), it is not always convenient to expect, let alone require, that one has equality in (6.19). Instead, what happens often, and turns out to be satisfactory in applications, is a weaker form of (6.5), namely, one in which $\widetilde{T}(X)$ can be proved to be large enough.

It is important to note that the spaces Y for which nonlinear PDEs are now solved by Theorem 3 include many highly discontinuous functions on Ω (see [OR, pp. 74-93]).

What is particularly interesting is that, in view of (6.19), a large variety of linear and nonlinear PDEs can be solved, in spite of the fact that the respective PDEs are known not to have solutions in distributions or in Sobolev spaces. Among such PDEs is the celebrated 1957 Hans Lewy impossibility example (6.1). In this regard, it was for the first time in [OR, chap. 6, 8] that this Hans Lewy example of a PDE not solvable in distributions or Sobolev spaces was nevertheless solved (through the method of order completion).

The correspondence between the solutions obtained in (6.19) and the usual classical solutions, (whenever the nonlinear PDEs in (6.2) may have classical solutions) follows easily from the way the commutative diagrams (6.6) are constructed. In other words, whenever the nonlinear PDEs in (6.2) happen to have classical solutions $U \in \mathcal{C}^m(\Omega)$, then they are also solutions in the sense of (6.19).

Recently, significant further improvements of the regularity of solutions were obtained through a refinement of the order completion method, [Wa1, Wa2, Wa3, Wa4]. The respective results indicate that the order completion method has considerable potential in attaining stronger regularity results than are currently known even without the use of functional analytic methods.

As far as the generality of the existence result of solutions, this was already attained to such an extent in [OR] that, at present, there appears to be no need for further extensions.

Finally, let us mention that the order completion method turns out to be significantly more powerful in solving large classes of nonlinear PDEs than the earlier introduced **nonlinear algebraic** method.[8]

Indeed, while the earlier nonlinear algebraic method can solve large classes of smooth linear or nonlinear PDEs, it falls short, even if not by much, in overcoming the Hans Lewy impossibility result (6.1). There are two main shortcomings of that algebraic method. First, the nonlinear PDEs which it can solve are significantly less general than those solved by the order completion method. Second, the solutions delivered by the algebraic method tend to have rather weak regularity properties, since they are given by generalized functions which are in spaces far larger than the Schwartz distributions or the Sobolev spaces. This is a sharp contrast with solutions delivered by the order completion method: solutions which are Hausdorff continuous functions.

Having said this, of course, it is important to note that the algebraic method in solving nonlinear PDEs is powerful enough to offer the first complete solution of Hilbert's Fifth Problem (see [Ro15]).

6.5 Appendix

6.5.1 Order Completion

A given poset $(X, \leq)$ is called **order complete** if and only if $\sup A, \inf A \in X$, for every $A \subseteq X$. If $\sup A \in X$ (respectively, $\inf A \in X$), only for every upper, (respectively, lower) bounded $A \subseteq X$, then $(X, \leq)$ is called **Dedekind order complete**. Clearly, $\mathbb{R}$ with its usual order is Dedekind order complete but not also order complete. On the other hand, the extended real line $\overline{\mathbb{R}} = [-\infty, \infty]$, as well as the closed intervals $[a, b] \subset \mathbb{R}$ are both Dedekind order complete and order complete.

[8] See [Ro1]–[Ro17], Zbl717*35001, MR92d:46098, MR89g:35001, Bull.AMS, Jan.1989, 96-101, and also subject 46F30 at `http://www.ams.org/msc/46Fxx.html`

Given two posets $(X, \leq)$ and $(Y, \leq)$, a mapping $\psi : X \to Y$ is called an **order isomorphic embedding** if and only if, for $x, x' \in X$, we have $x \leq x' \iff \psi(x) \leq \psi(x')$. If in addition ψ is also surjective, then it is called an **order isomorphism**.

The fundamental result with respect to order completion was obtained in 1937 by MacNeille: it states that for every poset $(X, \leq)$ which does not have a smallest or a largest element, there is an order complete poset $(\widetilde{X}, \leq)$ in which X is order-dense, that is, for every $\widetilde{x} \in \widetilde{X}$, there exists a subset $A \subseteq X$, such that $\widetilde{x} = \sup A$, [Ma] (see also [Lu] or [OR, Appendix]). Furthermore, $(\widetilde{X}, \leq)$ is *unique* up to **order isomorphism**.

A remarkable fact about MacNeille's result is that the order completion $\widetilde{X}$ is obtained in a manner which is a direct generalization of the construction of $\mathbb{R}$ from $\mathbb{Q}$ by Dedekind cuts. Thus MacNeille's method is called the **Dedekind order completion of** $(X, \leq)$, although it delivers an order complete poset $(\widetilde{X}, \leq)$, rather than a Dedekind order complete poset.

6.5.2 Hausdorff Continuous Functions

Let us denote by $\mathbb{A}$ the set of all functions $f : \mathbb{R} \ni x \mapsto [a, b]$, where $-\infty \leq a \leq b \leq \infty$. Thus such functions have **closed interval** values, and the respective intervals can be infinite at one, or at both ends. Usual or extended real valued functions $f : \mathbb{R} \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$, can naturally be seen as particular cases of such interval valued functions, if we consider them as having values at $x \in \mathbb{R}$, as given by the intervals $[f(x), f(x)]$ reduced to single points.

Now to every function $f \in \mathbb{A}$ we associate two functions $If, Sf : \mathbb{R} \to \overline{\mathbb{R}}$, defined for $x \in \mathbb{R}$, as follows:

$$If(x) = \sup_{V \in \mathcal{V}_x} \inf\{z \in f(y) \mid y \in V\},$$
$$Sf(x) = \inf_{V \in \mathcal{V}_x} \sup\{z \in f(y) \mid y \in V\},$$

where $\mathcal{V}_x$ is the set of neighborhoods of x.

Lastly, we also associate to f the function $Ff \in \mathbb{A}$, defined for $x \in \mathbb{R}$, by

$$Ff(x) = [If(x), Sf(x)].$$

Then an interval valued function $f \in \mathbb{A}$ is called **Hausdorff continuous**, if and only if it satisfies the following minimality condition: for every function $g \in \mathbb{A}$, we have for all $x \in \mathbb{R}$ such that $g(x) \subseteq f(x)$

$$Fg(x) = f(x).$$

We denote by $\mathbb{H}$ the set of all Hausdorff continuous functions.

Surprisingly, Hausdorff continuous functions have many of the important properties of usual continuous functions. For instance, if $f, g \in \mathbb{H}$ and if A is a *dense* subset of $\mathbb{R}$, then $f = g$ on A implies $f = g$ on $\mathbb{R}$.

As for the discontinuities of Hausdorff continuous functions, the following property is fundamental. Let $f \in \mathbb{H}$. Then we define $\underline{f}(x), \overline{f}(x)$ such that

$$f(x) = [\underline{f}(x), \overline{f}(x)],$$

where $\underline{f}, \overline{f} : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$, and $\underline{f}(x) \leq \overline{f}(x)$, for $x \in \mathbb{R}$. Let us now consider the set

$$\Gamma_f = \{x \in \mathbb{R} \mid \underline{f}(x) < \overline{f}(x)\},$$

that is, the points $x \in \mathbb{R}$ where the value of the function f is a genuine interval rather than a real or extended real number. Then it can be shown that Γ_f is meager.

Such regularity properties of Hausdorff continuous functions are particularly important in the context of solving PDEs though the order completion method. Obviously, the above definition of Hausdorff continuous functions can be extended to functions defined on any topological space with suitable properties, and thus in particular, to any open set in Euclidean space.

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