FACULTY FEATURE ARTICLE

6
Twisting with Fibonacci

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Abstract

Determining when two links are equivalent is one of the central goals of knot theory. This paper describes the Conway polynomial, a link invariant that offers one approach to this problem. When calculating the Conway polynomial of the $(n, 2)$ torus knots, we encounter the familiar patterns of Pascal's triangle and the Fibonacci sequence.

6.1 Introduction

Pick up a piece of string. Tangle it up, twist it around, knot it up, and then attach the ends. The result is a mathematical knot. Suppose a friend does the same thing. Try to twist, stretch, or otherwise deform the two tangled loops, without cutting the strings, so that they look exactly the same. If this is possible, then the knots are said to be equivalent. Determining whether or not two knots are equivalent is one of the central questions in knot theory.

Mathematically, a **knot** is a continuous closed curve in space that does not intersect itself. A **link** is the disjoint union of finitely many knots, where the number of components of the link is determined by the number of knots. If each component is assigned a direction, then the link is **oriented**. Two oriented links are equivalent if one can be deformed into the other in such a way that the orientation is preserved. A two-dimensional picture of a link is called a **projection.**

Figure 6.1: Do these two oriented link projections represent equivalent links?

Knot theorists use **link invariants** to distinguish between different links. If two links are equivalent, then calculating the link invariant for each link produces the same result, despite the fact that the link projections may appear to be drastically different.

One example of a link invariant is the **Conway polynomial**. Given a projection of an oriented link L, we can assign a polynomial $\nabla(L)$, described using the variable z. The polynomial is defined

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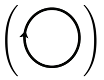
so that no matter how the link is twisted around in space, the polynomial will not change—any two projections of the same link will have the same Conway polynomial.

The Conway polynomial of a particular projection is calculated recursively by applying the following two definitions.

Definition 1. If L is equivalent to a single unknotted circle ("the unknot"), then its Conway polynomial is equal to 1; that is,

∇  $= 1.$ (6.1)

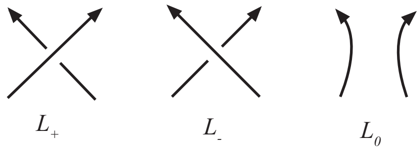
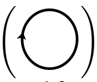


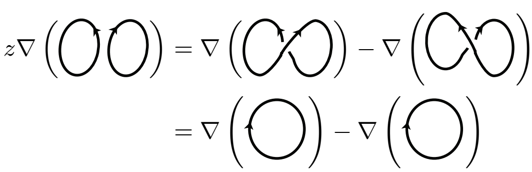
Figure 6.2: These three link projections are identical outside of the region shown.

Definition 2. Suppose L_+, L_-, and L_0 are three oriented link projections that are identical except near one crossing of L_+, where they appear as in Figure 6.2. Then their Conway polynomials satisfy the relation

$$z\nabla(L_0) = \nabla(L_+) - \nabla(L_-). \tag{6.2}$$

These two defintions, together with the requirement that two projections of the same link must have the same Conway polynomial, suffice for calculating the Conway polynomial of any knot or link. We can always unknot a link by changing finitely many crossings. By applying Definition 2 at one of these crossings and appropriately assigning the roles of L_+, L_-, and L_0, we can find the Conway polynomial of a given link in terms of the Conway polynomials of links with fewer crossings. We can eliminate all crossings by repeating this process, resulting in only the trivial knot or trivial links.

By Definition 1, we know ∇  $= 1$. The Conway polynomial of a trivial link is 0. To see this, label the trivial link as L_0 and form L_+ and L_- by joining two of the components using a positive and negative crossing respectively. Applying (6.2) gives



$= 0.$

The following example illustrates how these two definitions are used to calculate the Conway polynomial of the trefoil knot.

Example 3. The Conway polynomial of the trefoil knot is $1 + z^2$.

Proof. Select one crossing in the trefoil knot. Since the crossing is a positive crossing, label the trefoil knot as L_+ and replace the region near the chosen crossing as shown in Figure 6.2 to obtain

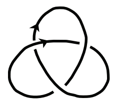
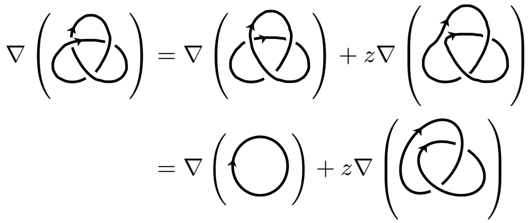
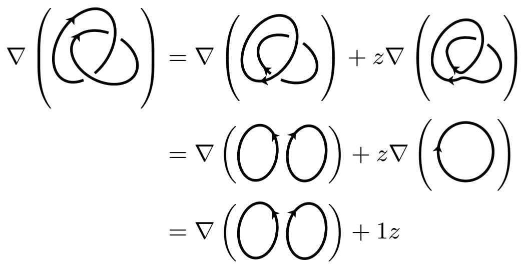


Figure 6.3: The trefoil knot.

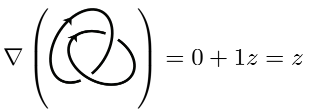
L_- and L_0. Then use (6.2).



The Conway polynomial of the unknot is 1. To find the Conway polynomial of the link, select another crossing and apply (6.2) again.



Since the Conway polynomial of the trivial link is 0, we see that



and the Conway polynomial of the trefoil is $1 + z(z) = 1 + z^2$, as claimed. □

Before continuing, we urge the reader to try a problem or two.

Exercise 4. A different projection of the trefoil knot is shown in Figure 6.4. Make this knot out of string and manipulate the string to show that this knot is equivalent to the one used in Example 3. Verify that calculating the Conway polynomial starting with this projection also results in $1 + z^2$.

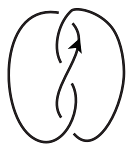


Figure 6.4: Another projection of the trefoil knot

Exercise 5. Show that the Conway polynomial of the figure-eight knot, shown in Figure 6.5, is $1 - z^2$.

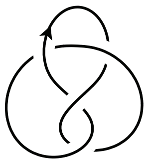


Figure 6.5: The figure-eight knot.

6.2 Torus Links

Suppose you have two strings, side by side, and the tops of the strings are fixed. Twist the right string over the left string n times, and orient both strings in the same direction. Without introducing any additional crossings, attach the bottoms of the strings to the tops, as shown in Figure 6.6. The resulting knot or link is known as an $(n, 2)$ torus link, because it lies flat on a torus, wrapping around twice longitudinally and n times through the center.[1]

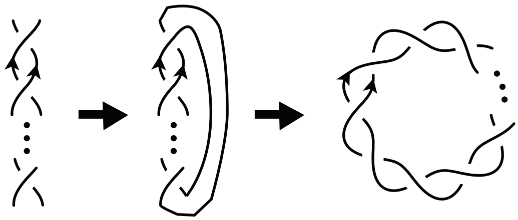


Figure 6.6: An $(n, 2)$ torus link.

Now we explore what happens when we calculate the Conway polynomials of these links. Let $\mathcal{L}_n$ denote the $(n, 2)$ torus link. Notice that $\mathcal{L}_1$ is the unknot, $\mathcal{L}_2$ is the link from Example 3, and $\mathcal{L}_3$ is the trefoil. When n is odd, $\mathcal{L}_n$ is a knot. When n is even, $\mathcal{L}_n$ is a two-component link. Furthermore, for $n \geq 3$, changing a single crossing in the $(n, 2)$ torus link results in a projection of the $(n - 2, 2)$ torus link. See Figure 6.7.

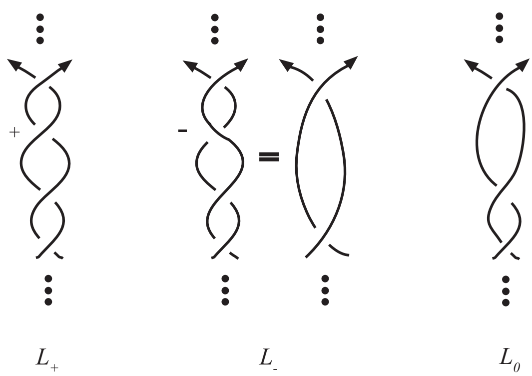


Figure 6.7: Applying (6.2) to torus links with $L_+ = \mathcal{L}_n$, $L_- = \mathcal{L}_{n-2}$, and $L_0 = \mathcal{L}_{n-1}$.

[1] See [Ad, Section 5.1] for a general description of (n, m) torus knots and links.

Using the definitions for calculating the Conway polynomial, we observe that for all $n \geq 3$,

$$\nabla(\mathcal{L}_n) = \nabla(\mathcal{L}_{n-2}) + z\nabla(\mathcal{L}_{n-1}). \tag{6.3}$$

This quickly leads to a table of Conway polynomials:

$$\begin{aligned}
\nabla(\mathcal{L}_1) &= 1 \\
\nabla(\mathcal{L}_2) &= z \\
\nabla(\mathcal{L}_3) &= 1 + z^2 \\
\nabla(\mathcal{L}_4) &= 2z + z^3 \\
\nabla(\mathcal{L}_5) &= 1 + 3z^2 + z^4 \\
\nabla(\mathcal{L}_6) &= 3z + 4z^3 + z^5 \\
\nabla(\mathcal{L}_7) &= 1 + 6z^2 + 5z^4 + z^6 \\
\nabla(\mathcal{L}_8) &= 4z + 10z^3 + 6z^5 + z^7 \\
\nabla(\mathcal{L}_9) &= 1 + 10z^2 + 15z^4 + 7z^6 + z^8 \\
\vdots \qquad & \qquad \vdots
\end{aligned}$$

6.3 Pattern Recognition and Formulas

If we look at the table of Conway polynomials, we can immediately make a few observations. First, the degree of the polynomial for the $(n, 2)$ torus link is $n - 1$. The polynomials are all **monic**, meaning that the highest order term has 1 as its coefficient. When n is odd, the constant term of the polynomial is 1 and the polynomial contains only even powers of z. When n is even, the polynomial contains only odd powers of z, and the coefficient of the z term is $n/2$.

These observations begin to reveal the behavior of the Conway polynomial of the $(n, 2)$ torus link, but we can do better by taking a closer look at the pattern formed by all the coefficients:

$$\begin{array}{lcccccccccc}
\nabla(\mathcal{L}_1) = 1 \\
\nabla(\mathcal{L}_2) = & 1z \\
\nabla(\mathcal{L}_3) = 1 & + & 1z^2 \\
\nabla(\mathcal{L}_4) = & 2z & + & 1z^3 \\
\nabla(\mathcal{L}_5) = 1 & + & 3z^2 & + & 1z^4 \\
\nabla(\mathcal{L}_6) = & 3z & + & 4z^3 & + & 1z^5 \\
\nabla(\mathcal{L}_7) = 1 & + & 6z^2 & + & 5z^4 & + & 1z^6 \\
\nabla(\mathcal{L}_8) = & 4z & + & 10z^3 & + & 6z^5 & + & 1z^7 \\
\nabla(\mathcal{L}_9) = 1 & + & 10z^2 & + & 15z^4 & + & 7z^6 & + & 1z^8
\end{array}$$

Notice the appearance of Pascal's triangle along the diagonals of the Conway polynomial coefficients! Alternatively, we can find these coefficients of the Conway polynomials within Pascal's triangle, as seen in Figure 6.8.

Recall that the entries in Pascal's triangle are the binomial coefficients. This suggests the following formulae.

Theorem 6. *The Conway polynomial of the* $(2n+1, 2)$ *torus knots is given by the equation*

$$\begin{aligned}
\nabla(\mathcal{L}_{2n+1}) &= \binom{n}{0} + \binom{n+1}{2} z^2 + \binom{n+2}{4} z^4 + \cdots + \binom{2n}{2n} z^{2n} \\
&= \sum_{j=0}^{n} \binom{n+j}{2j} z^{2j}
\end{aligned} \tag{6.4}$$

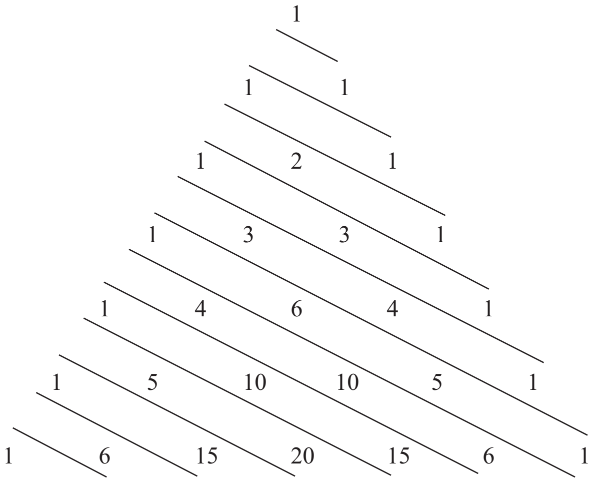


Figure 6.8: The coefficients of the Conway polynomials of the $(n, 2)$ torus knots and links can be found within Pascal's triangle.

and the Conway polynomial of the $(2n, 2)$ *torus links is given by*

$$\begin{aligned}\nabla(\mathcal{L}_{2n}) &= \binom{n}{1}z + \binom{n+1}{3}z^3 + \binom{n+2}{5}z^5 + \cdots + \binom{2n-1}{2n-1}z^{2n-1} \\ &= \sum_{j=0}^{n-1}\binom{n+j}{2j+1}z^{2j+1}. \end{aligned} \tag{6.5}$$

Proof. We prove (6.4) and (6.5) using the principle of mathematical induction. Since $\mathcal{L}_1$ is the unknot, we know that $\nabla(\mathcal{L}_1) = 1 = \binom{0}{0}$. In Example 3, we showed that $\nabla(\mathcal{L}_2) = z = \binom{0}{0}z$, so the results are valid when $n = 0$.

Assume we know the formulae hold for all positive integers less than m. We prove the formula holds when $m = 2n$ is even. (The case for m odd is similar, and is left to the reader.) We have

$$\begin{aligned}\nabla(\mathcal{L}_{2n}) &= \nabla(\mathcal{L}_{2n-2}) + z\nabla(\mathcal{L}_{2n-1}) \\ &= \sum_{j=0}^{n-2}\binom{n-1+j}{2j+1}z^{2j+1} + z\sum_{j=0}^{n-1}\binom{n-1+j}{2j}z^{2j} \\ &= \sum_{j=0}^{n-2}\left(\binom{n-1+j}{2j+1} + \binom{n-1+j}{2j}\right)z^{2j+1} + z^{2n-1} \\ &= \sum_{j=0}^{n-1}\binom{n+j}{2j+1}z^{2j+1}. \end{aligned}$$

□

The sudden appearance of Pascal's triangle allowed us to conjecture and prove a result about the Conway polynomials of mathematical knots. As frequently occurs in mathematics, results from a seemingly unrelated field can be utilized where least expected.

6.4 Torus Links and the Fibonacci Sequence

In [Ka], Kauffman observed another relationship: If we evaluate the Conway polynomials of these torus links at $z = 1$, then we obtain the Fibonacci sequence.

Recall that the Fibonacci sequence $\{f_n\}$ is defined by the following recursive relation.

$$f_1 = 1$$
$$f_2 = 1$$
$$f_n = f_{n-2} + f_{n-1}$$

When $z = 1$, we see that $\nabla(\mathcal{L}_1)|_{z=1} = 1$, $\nabla(\mathcal{L}_2)|_{z=1} = 1$, and the recursive relation from (6.3) becomes

$$\nabla(\mathcal{L}_n)\big|_{z=1} = \nabla(\mathcal{L}_{n-2})\big|_{z=1} + 1 \cdot \nabla(\mathcal{L}_{n-1})\big|_{z=1},$$

which establishes the identity

$$\nabla(\mathcal{L}_n)\big|_{z=1} = f_n.$$

Combining this with (6.4) and (6.5), we obtain the identities

$$f_{2n+1} = \sum_{j=0}^{n} \binom{n+j}{2j} \tag{6.6}$$

$$f_{2n} = \sum_{j=0}^{n-1} \binom{n+j}{2j+1}. \tag{6.7}$$

The fact that the Fibonacci numbers occur as sums of the diagonals of Pascal's triangle shown in Figure 6.8 was discovered by Edouard Lucas in 1876. See [Ko] for a direct proof.

6.5 Conclusion

Since the polynomials given in (6.4) and (6.5) are distinct, we know that the $(n, 2)$ torus knots and links are distinct for different values of n. The Conway polynomial can be a useful tool for telling many knots and links apart, including those in Figure 6.1.

Exercise 7. Use the Conway polynomial to prove that the oriented link projections in Figure 6.1 do *not* represent equivalent links.

The Conway polynomial is equivalent to a normalized version of the very first polynomial for knots and links, which was invented by J. Alexander in 1928. Alexander defined his original polynomial using the **Seifert matrix** constructed from a surface spanning an oriented link, and in 1969 John Conway discovered the polynomial could be derived more simply using the above definitions. The Alexander-Conway polynomial was one of the major tools used for distinguishing knots for the next 60 years.

However, the Conway polynomial cannot distinguish between all knots or links. Two knots or links can have the same Conway polynomials even if they are not equivalent. For example, **splittable links** are links that can be deformed so that the components lie on different sides of a plane in $\mathbb{R}^3$.

Exercise 8. Show that the Conway polynomial of any splittable link is 0. (Hint: Label the splittable link as L_0.)

Another example which illustrates the limitations of the Conway polynomial is the 11-crossing Conway knot, shown in Figure 6.9. This knot has Conway polynomial 1, even though it cannot be unknotted.

In fact, given any knot, there are infinitely many other knots with the same Conway polynomial (*cf.* [Cr, p. 164]).

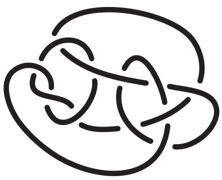


Figure 6.9: This 11 crossing knot has the same Conway polynomial as the unknot.

In 1984, Vaughan Jones discovered a connection between **von Neumann algebras** and **braid groups**, which led to a new polynomial for knots and links. The **Jones polynomial** has the advantage that there are no known examples of knots that have the same Jones polynomial as the trivial knot. Jones' discovery encouraged other mathematicians to search for more knot polynomials. This quickly led to the discovery of the **Homfly polynomial**, a two-variable generalization of both the Alexander-Conway and Jones polynomials which was developed independently by Jim Hoste, Adrian Ocneanu, Raymond Lickorish and Ken Millett, Peter Freyd and David Yetter, and Jozef Przytycki and Pawel Traczyk. Still other polynomial invariants grew out of a surprising connection between knot theory and theoretical physics.

The connections between knot polynomials and previously unrelated fields of mathematics led to renewed interest in the mathematical theory of knots and links, and rapid advances in the subject. Still, none of these polynomials provides a complete invariant—infinitely many examples exist of pairs of non-equivalent knots that still have the same knot polynomials. Knot theorists continue to search for more ways to distinguish knots and links.

We refer the interested reader to [Ad, Cr, Sc] for more about knot theory and to [BQ, En] for additional properties of Pascal's triangle and the Fibonacci sequence. We conclude with a final problem.

Take a rubber band, and twist it n times while holding onto two ends of the rubber band. Link the ends together using a clasp with two crossings so that the resulting knot is alternating. The result is a **twist knot**, pictured in Figure 6.10. Note that the figure-eight knot in Figure 6.5 is an example of a twist knot.

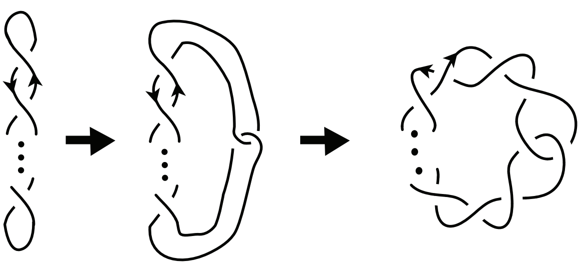


Figure 6.10: A twist knot

Exercise 9. Find a formula for the Conway polynomials of the twist knots. Use this to conclude that the twist knots are distinct for different values of n, and that none of the twist knots are equivalent to $(n, 2)$ torus knots.

6.6 Acknowledgments

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