FEATURE

12
Solutions

Projective Paranoia

S07 – 3. The incircle Ω_{ABC} of a triangle ABC is tangent to BC, CA, AB at P, Q, R respectively. Rays PQ and BA intersect at M, rays PR and CA intersect at N, and the incircle Ω_{MNP} of triangle MNP is tangent to MN and NP at X and Y respectively. Given that X, Y and B are collinear, prove:

(a) Circles Ω_{ABC} and Ω_{MNP} are congruent, and

(b) these circles intersect each other in 60° arcs.

Proposed by Zachary Abel '10.

Solution by the proposer. Let $\operatorname{cr}(A_1, A_2; A_3, A_4)$ denote the cross ratio $\frac{A_1A_3/A_3A_2}{A_1A_4/A_4A_2}$ of four collinear points A_1, A_2, A_3, A_4. Let ρ and τ denote the polar maps through circles Ω_{ABC} and Ω_{MNP} respectively, and let I and J be the respective centers of these two circles.

Let Ω_{MNP} touch MP at Z, and define $MN \cap BC = S$. We first show that X, Z, and C are collinear. As $\rho(S) = AP$, it follows that $\operatorname{cr}(R, Q; S, RQ \cap AB) = -1$, and hence by perspectivity through A, $\operatorname{cr}(B, C; S, P) = -1$. An identical argument proves that $\operatorname{cr}(N, M, MN \cap YZ, X) = -1$. Letting $C' = XZ \cap BC$, we may calculate

$$\operatorname{cr}(B, C'; S, P) \overset{X}{=} \operatorname{cr}(Y, Z; MN \cap YZ, XP \cap YZ) \overset{P}{=} \operatorname{cr}(N, M; MN \cap YZ, X) = -1$$

where the notation $\overset{J}{=}$ indicates that equality follows by a perspectivity about point J. Since $\operatorname{cr}(B, C; S, P) = \operatorname{cr}(B, C'; S, P)$, it follows that $C = C'$, as claimed.

By Pascal's theorem on hexagon $BXCNPM$, we find that Y, A, and Z are collinear. Then by the converse of Brianchon's theorem on hexagon $RYNMZQ$, there must be some conic tangent to line RN at Y, tangent to QM at Z, and tangent to lines MN and QR. Such a conic is uniquely determined by the first three of these four facts and must therefore coincide with circle Ω_{MNP}, hence QR is tangent to this circle. Label this point of tangency T. By well-known properties of circumscribed quadrilaterals, lines XY, MR, and ZT are concurrent, *i.e.* T lies on line ZB. Likewise, T is collinear with Y and C.

We may apply similar arguments to circle Ω_{ABC}. Letting $U = CT \cap AB$ and $V = BT \cap AC$, the converse of Brianchon's Theorem proves that UV is tangent to circle Ω_{ABC} at some point W. Also as above, RW must pass through Z and QW passes through Y.

Note that

$$\rho(UC) = \rho(U) \cap \rho(C) = RW \cap PQ = Z$$

and likewise $\rho(VB) = Y$, so $\rho(T) = \rho(UC \cap VB) = YZ$. In particular, $IT \perp YZ$, and hence $IT \parallel PJ$. Similarly, $\tau(A)$ is the join of $\tau(NQ) = XY \cap TZ = B$ and $\tau(MR) = XZ \cap YT = C$, hence $JA \perp BC$. Let line JA meet BC and the top of circle Ω_{MNP} at D and E respectively.

By Pascal's theorem on hexagon $PPQQRR$ in circle Ω_{PQR}, point S lies on line QR. As $\angle JTS = \angle JDS = 90^\circ$, quadrilateral $JTDS$ is cyclic. Using α, β, and γ for the angles of $\triangle ABC$, we may calculate

$$\angle JET = \tfrac{1}{2}\angle DJT = \tfrac{1}{2}\angle DST = \tfrac{1}{2}\left(180^\circ - \angle RQP - \angle QPS\right) = 45^\circ - \tfrac{\alpha}{4} - \tfrac{\gamma}{2}$$

and, with $YZ \cap BC = F$,

$$\angle PFZ = \angle CPZ - \angle FZP = (90^\circ - \tfrac{\gamma}{2}) - (45^\circ + \tfrac{\alpha}{4}) = \angle JET.$$

This is enough to conclude that $ET \perp YZ$, *i.e.* that ETI are collinear and furthermore lie on a line parallel to PJ. Thus, $JEIP$ is a parallelogram, and as long as it is not degenerate, we may conclude that $|JE| = |IP|$, *i.e.* that $\Omega_{ABC} \cong \Omega_{PQR}$.

If it is degenerate, then triangle ABC must be isosceles with A at its vertex. In this case, line MN is parallel to BC, so from $XNY \sim BPY$ we find that $|BP| = |PY|$. And since $BI \perp PY$ and $BP \perp PJ$, it follows that right triangles BPI and PYJ are congruent. Thus $|PI| = |YJ|$, and again the two circles are congruent.

Let r be the common radius length. To prove part (b), it suffices to show that $d = |IJ| = r\sqrt{3}$. Consider the inversion ι through Ω_{MNP}. The points $\iota(P)$, $\iota(Q)$, and $\iota(R)$ correspond to the midpoints of ZY, YT, and TZ respectively, so $\iota(\Omega_{ABC})$ is the nine-point-circle of triangle YTZ. In particular, the radius of $\iota(\Omega_{ABC})$ is half the radius of Ω_{MNP}, *i.e.*, $\frac{r}{2}$.

Let IJ intersect Ω_{ABC} at H and K, so that $|JH| = d - r$ and $|JK| = d + r$. Then $|\iota(HK)| = |\iota(JH)| - |\iota(JK)| = \frac{r^2}{d-r} - \frac{r^2}{d+r}$. But $\iota(HK)$ is a diameter of $\iota(\Omega_{ABC})$, and therefore has length r. This indeed gives $d = r\sqrt{3}$, as needed for part (b). $\square$

Euler and Napoleon

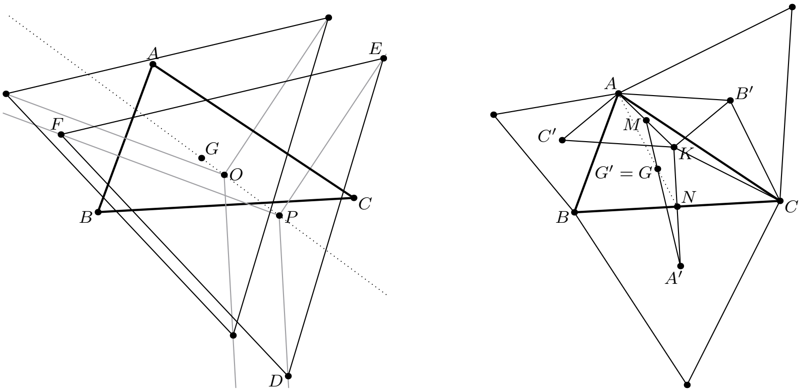
F07 – 1. Consider $\triangle ABC$ an arbitrary triangle and P a point in its plane. Let D, E, and F be three points on the lines through P perpendicular to the lines $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Prove that if $\triangle DEF$ is equilateral and if P lies on the Euler line of $\triangle ABC$, then the center of $\triangle DEF$ also lies on the Euler line of $\triangle ABC$.

Proposed by Cosmin Pohoata (Bucharest, Romania) and Darij Grinberg (Germany).

Solution by Yasuhide Minoda (Tetsu Ryoku-Kai, Japan). Let O be the circumcenter of $\triangle ABC$. By parallel translation, it suffices to consider the case $P = O$ (see Figure 12.1(a)).

Let A', B' , C' be centers of equilateral triangles outside of $\triangle ABC$, drawn on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, respectively. It is well known that $\triangle A'B'C'$ is equilateral (the outer Napoleon triangle). [***Editor's note:*** if $\triangle ABC$ and $\triangle DEF$ have opposite orientation, we should instead take A', B', and C' as the centers of equilateral triangles drawn *inward* on the three sides of $\triangle ABC$, so that $\triangle A'B'C'$ is the *inner* Napoleon triangle, which is also equilateral.]

Lemma 1. *$\triangle A'B'C'$ and $\triangle DEF$ are similar with respect to O. In particular, the center of $\triangle A'B'C'$, the center of $\triangle DEF$ and O are collinear.*



(a) Without loss, we may assume $P = O$.

(b) The centroids of $\triangle ABC$ and $\triangle A'B'C'$ coincide.

Figure 12.1: Figures for Problem F07 – 1.

Proof. Without loss of generality, we can assume $\angle BAC \neq 2\pi/3$. Scale up or down $\triangle DEF$ with respect to O to form $\triangle D'E'F'$ so that D' coincides with A'. We want to show $E' = B'$ and $F' = C'$.

Rotate line $\overline{OB'}$ by $\pi/3$ [or $-\pi/3$ depending on orientation] around A' and denote it by ℓ. B' and E' are moved on ℓ by this rotation. On the other hand, B' and E' are moved to C' and E', respectively, and these points are both on line $\overline{OC'}$. Thus, F' and C' must be the intersection of line $\overline{OC'}$ and ℓ (note that these lines intersect at one point because $\angle BAC \neq 2\pi/3$).

Therefore, $C' = F'$ and $B' = E'$. It follows that $\triangle A'B'C'$ and $\triangle DEF$ are similar with respect to O. □

Lemma 2. *The center of* $\triangle A'B'C'$ *coincides with the centroid of* $\triangle ABC$.

Proof. Let K be a point symmetric to A' with respect to line $\overline{BC}$ (see Figure 12.1(b)). Triangles $\triangle B'KC$ and $\triangle ABC$ are similar with a ratio $1 : \sqrt{3}$. In addition, $\overline{AC'} : \overline{AB} = 1 : \sqrt{3}$. Thus, $\overline{AC'} = \overline{B'K}$. Similarly, $\overline{AB'} = \overline{C'K}$. So $AC'KB'$ is a parallelogram.

Let the center of $AC'KB'$ be M, and the midpoint of $\overline{BC}$ be N. The centroid G' of $\triangle A'B'C'$ lies on $\overline{MA'}$ and $\overline{MG'} : \overline{G'A'} = 1 : 2$. Thus,

$$\frac{\overline{A'N}}{\overline{NK}} \cdot \frac{\overline{KA}}{\overline{AM}} \cdot \frac{\overline{MG'}}{\overline{G'A'}} = \frac{1}{1} \cdot \left(-\frac{2}{1}\right) \cdot \frac{1}{2} = -1.$$

So, by Menelaus' theorem, G' lies on the median $\overline{AN}$ of $\triangle ABC$. Similarly, G' also lies on other medians of $\triangle ABC$. The lemma is proved. □

By Lemma 1 and Lemma 2, the centroid of $\triangle DEF$ is on line $\overline{OG}$, namely the Euler line of $\triangle ABC$. □

Also solved by the proposers.

Dastardly Haberdashery

F07 – 2. Professor Perplex has rounded up his $n > 0$ hat-game seminar students and made the following ominous announcement:

> "I have assigned each of you a hat according to a uniform probability distribution, which I will put on your head after allowing you time to discuss a strategy. Hats come in $h > 0$ different colors, but some colors might be reused and others might not be used at all. Each student will be given a list of the h colors. Nobody will be able to see his or her own hat, but everyone will have the opportunity to observe all the other hats. Then, you will all be instructed to simultaneously write down one of the colors. If any student correctly identifies the color of his or her own hat, then there will be no final exam this semester. Otherwise, I will assign a week-long haberdashery final."

What is the probability that the students have to take a final, assuming best play?

Proposed by John Hawksley (Massachusetts Institute of Technology '08) and Scott D. Kominers '09.

Solution by Charlie Pasternak (Takoma Park Middle School). A student cannot deduce any information about his own hat color from what he sees alone. This means that the probability of a student making a correct guess, and the expected number of correct guesses over time, remains constant, so the students' best strategy is to spread out the correct guesses as thinly as possible to "waste" as few as possible.

This best strategy is: if $n \geq h$, take h students and assign them a numbers $\{0, 1, \ldots, h-1\}$, and if $n < h$, assign the students the numbers $\{0, \ldots, n-1\}$. Then, assign each color a number from 0 to $h-1$. When the hats are put on, each student sums up the colors of the hats seen, then writes for his own hat color the color corresponding to the difference between his assigned number and the sum of the hats he sees, modulo h.

If a student's assigned number is the sum modulo h of the colors, his guess will be correct. If $n \geq h$, at least one person's assigned number is the sum modulo h of the colors, so at least one guess will be right. If $n < h$, then the chance that one of the students' assigned numbers is the sum modulo h of the colors is $\frac{n}{h}$, as is the chance of a correct guess.

Therefore, the chance of the students taking a final is $\max(0, 1 - \frac{n}{h})$. □

Also solved by Sherry Gong '11, Arnav Tripathy '11, Ray C. He (Massachusetts Institute of Technology '07), and the proposers.

Restricted Roots' Radii

F07 – 3. Find all integer monic polynomials $f(x)$ such that

(i) $f(x) = f(1 - x)$ and

(ii) all complex zeros of f lie in the disk $|z| < \sqrt[5]{2}$.

Proposed by Vesselin Dimitrov '09.

Solution by Noam D. Elkies (Harvard University). The polynomials $f_0(x) = x^2 - x$ and $f_1(x) = x^2 - x + 1$ satisfy both conditions (the latter has roots of absolute value 1 at the primitive sixth roots of unity). Therefore so does $f_0(x)^{a_0} f_1(x)^{a_1}$ for any nonnegative integers a_0 and a_1. We claim that these are the only such polynomials, indeed the only polynomials satisfying (i) whose roots lie in $|z| < r := 1.3$ (note that $2^{1/5} < 1.15 < r$).

Let f be any polynomial satisfying both conditions, and let α be a complex zero of f. Then $1 - \alpha$ is also a complex zero of f, so α lies in the intersection of the open discs of radius r about 0 and 1. We claim:

Lemma 3. *If z is a complex number such that $|z| < r$ and $|1 - z| < r$, then $|f_0(z)^2 f_1(z)^3| < 1$.*

Assuming this lemma, it follows that the algebraic integer $y := f_0(\alpha)^2 f_1(\alpha)^3$ has the property that y and all its conjugates (which are also values of $f_0^2 f_1^3$ at roots of f) have absolute value less than 1. But then the norm of y, which is the product of those conjugates, is a rational integer of absolute value less than 1. Therefore y is an algebraic number of norm zero, whence $y = 0$. This means that α is a root of either f_0 or f_1. Moreover, for each $j = 0, 1$ the two roots of f_j are switched by the involution $x \leftrightarrow 1 - x$ of $\mathbb{C}$, and thus have the same multiplicity as complex zeros of f. Letting a_j be this common multiplicity, we find that $f = f_0(x)^{a_0} f_1(x)^{a_1}$, as claimed.

We prove the lemma via the following explicit calculation. Let R be the intersection of the closed discs $|z| \leq r$ and $|1 - z| \leq r$. Let $y(z) := f_0(z)^2 f_1(z)^3$. We claim $|y(z)| \leq 1$ for all $z \in R$. The function y is analytic; hence by the maximum principle it suffices to prove $|y(z)| \leq 1$ on the boundary of R. By symmetry about the real axis and the vertical line $\mathrm{Re}(z) = 1/2$, we may assume $z = x + i\sqrt{r^2 - x^2}$ for some $x \in [1/2, r]$. For lack of a better idea, we expand $|y(z)|^2$, obtaining a polynomial $P_8(x) \in \mathbb{Q}[x]$ of degree 8. A numerical plot suggests that this polynomial does not exceed 0.9 on $[1/2, r]$. Since $P_8(1/2) = r^8(1 - r^2)^6 < 0.9$, we can prove that $P_8(x) < 0.9$ for all $x \in [1/2, r]$ by checking that this interval contains no roots of $P_8 - 0.9$, and this is confirmed using Sturm's method (implemented in `gp` as `polsturm`). This completes the proof of the lemma and the solution of F07–3.

Remark. We could likewise use $f_0(\alpha)^{b_0} f_1(\alpha)^{b_1}$ for any positive integers b_0, b_1. The simple choice $b_0 = b_1 = 1$ suffices to solve problem F07–3 as stated, but would only let us improve r from about 1.15 to about 1.27. The choice $(b_0, b_1) = (2, 3)$ is not quite optimal either, but can be used for any r less than the positive root $r_0 = 1.304+$ of $x^{10} - 3x^8 + 3x^6 - x^4 - 1$ (for which $f_0(z)^2 f_1(z)^3 + 1$ has roots with $|z| = |1 - z| = r_0$), and numerical experimentation suggests that this r_0 is very nearly as far as this technique can be pushed. At any rate we cannot take $r_0 > 1.3503$ because the tenth-degree polynomial $f_0^2 f_1^3 + 1$ satisfies condition (i) and has a pair of complex roots of absolute value $1.35025542+$.

Remark. While we would get a worse bound had we used $(b_0, b_1) = (1, 1)$ instead of the exponents $(2, 3)$ in our Lemma, the proof would be a routine albeit tedious calculus exercise, because instead of the octic P_8 we would have only a cubic polynomial to maximize. This would still be enough to solve problem F07–3 as stated, or even with the bound $2^{1/5}$ raised to $2^{1/3} = 1.2599+$. (The cutoff value r_0 would then be the positive root $1.272+$ of $x^4 - x^2 - 1$.) We do not know whether the proposer's choice of $2^{1/5}$ allows for a more elegant proof. □

Editor's note. Indeed, the proposer's solution uses a similar method with $b_0 = b_1 = 1$. His choice of $\sqrt[5]{2}$ allows for a clean proof of the requisite lemma: From the identity $5(x^2-x)(x^2-x+1) = x^5 + (1-x)^5 - 1$, the inequality

$$|f_0(\alpha)f_1(\alpha)| = \frac{1}{5}\left|\alpha^5 + (1-\alpha)^5 - 1\right| < \frac{1}{5}\left(|\alpha|^5 + |1-\alpha|^5 + 1\right) = 1$$

follows immediately.

Also solved by the proposer.

A Surprisingly Constant Limit

F07 – 4. Let $a, b \geq 0$ be two nonnegative numbers. Find the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k+b+\sqrt{n^2+kn+a}}.$$

Proposed by Ovidiu Furdui (University of Toledo).

Solution by Paolo Perfetti (Università degli studi di Tor Vergata Roma, Math. Dept.). The key is to observe that the limit in independent of a and b. In view of this fact we compute the limit taking $a = b = 0$, getting

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k+\sqrt{n^2+nk}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}+\sqrt{1+\frac{k}{n}}} = \int_0^1 \frac{1}{1+x+\sqrt{1+x}}\,dx$$
$$= \int_1^2 \frac{1}{t+\sqrt{t}}\,dt = 2\int_1^{\sqrt{2}}\frac{1}{1+y}\,dy = 2\ln\frac{1+\sqrt{2}}{2}.$$

Now we prove that the limit is independent of a and b. First of all we write

$$\frac{1}{r+b+\sqrt{nr+a}} = \frac{1}{r+b+\sqrt{nr}\sqrt{1+\frac{a}{nr}}} = \frac{1}{r+b+\sqrt{nr}(1+O(\frac{1}{nr}))}$$
$$= \frac{1}{r+b+\sqrt{nr}}\left(1+O\left(\frac{1}{nr}\right)\right)$$

and

$$\sum_{r=n+1}^{2n}\frac{O(\frac{1}{nr})}{r+b+\sqrt{nr}} = \sum_{r=n+1}^{2n} O\left(\frac{1}{nr^2}\right) = n\cdot O\left(\frac{1}{n^3}\right),$$

where the limit of the above expression as n approaches ∞ is zero. Thus,

$$\lim_{n\to\infty}\sum_{r=n+1}^{2n}\frac{1}{r+b+\sqrt{nr+a}} = \lim_{n\to\infty}\sum_{r=n+1}^{2n}\frac{1}{r+b+\sqrt{nr}}.$$

Similarly,

$$\sum_{r=n+1}^{2n}\frac{1}{r+b+\sqrt{nr}} = \sum_{r=n+1}^{2n}\frac{1}{(r+\sqrt{nr})(1+\frac{b}{r+\sqrt{nr}})} = \sum_{r=n+1}^{2n}\frac{(1+O(r^{-1}))}{r+\sqrt{nr}}$$

and

$$\lim_{n\to\infty}\sum_{r=n+1}^{2n}\frac{O(r^{-1})}{r+\sqrt{nr}}=\lim_{n\to\infty}\sum_{r=n+1}^{2n}O(r^{-2})=\lim_{n\to\infty}n\cdot O(n^{-2})=0,$$

so

$$\lim_{n\to\infty}\sum_{r=n+1}^{2n}\frac{1}{r+b+\sqrt{nr}}=\lim_{n\to\infty}\sum_{r=n+1}^{2n}\frac{1}{r+\sqrt{nr}}.$$

As this is independent of a and b, the proof is complete. □

Also solved by The Northwestern University Math Problem Solving Group and the proposer.