FACULTY FEATURE ARTICLE

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Tiling with Commutative Rings

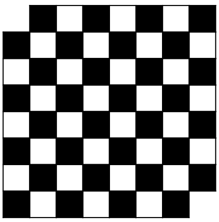
Thomas Lam[†]
Harvard University
Cambridge, MA 02138
tfylam@math.harvard.edu

Abstract

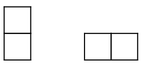
We explain an approach, due originally to Barnes, to tiling problems using some commutative algebra. We investigate in particular the occurence of coloring arguments in tiling problems. The only prerequisites are linear algebra and familiarity with rings and ideals.

5.1 A Recreational Problem

Consider the collection R of squares obtained from the chessboard by removing two opposite corners:



Can this configuration be covered with the vertical and horizontal dominoes



so that every square is covered by exactly one domino? In other words, can R be **tiled** by vertical and horizontal dominoes?

The coloring gives away the answer to this well-known problem. The region R has 32 black squares and 30 white squares. Since each domino covers exactly one black and one white square, no tiling is possible. The aim of this article is to explain a way to tackle tiling problems using a little commutative algebra. More precisely, we will explain how to obtain **coloring arguments**, similar to the above chessboard coloring, in a systematic way. I will assume that the reader is familiar with linear algebra and has seen rings and ideals before.

5.2 Tiles, Regions, and Tiling Problems

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the natural numbers. A **tile** or **region** is a finite subset of $\mathbb{N}^2$ considered as a collection of boxes in the first quadrant.[1] The tiling problems that we shall consider are of the following form: given a (possibly infinite) set $\mathbf{T}$ of tiles and a region R, can R be tiled (that is, covered with tiles so that each square in R is covered once)? Each tile $\tau \in \mathbf{T}$ can be translated anywhere within $\mathbb{N}^2$ and used as many times as desired but we shall insist that *rotations and reflections*

[†] Thomas Lam was born in Hong Kong and grew up in Australia. He earned his BSc in 2001 from University of New South Wales (Australia) and received his PhD in 2005 from MIT, studying with Richard Stanley. He has been Benjamin Peirce Assistant Professor at Harvard University since then.

[1] The interested reader will have no trouble generalizing our statements to higher dimensions.

are not allowed. If we want to allow rotations of a tile then they must be added to $\mathbf{T}$ separately. Because we may translate tiles as much as we like, we will also assume that each tile $\tau \in \mathbf{T}$ has been translated as far southwest as possible, so that it touches the x- and y-axes. Thus, in the above chessboard problem, $\mathbf{T}$ consists of two elements: the vertical domino $V = \{(0,0),(0,1)\}$ and horizontal domino $H = \{(0,0),(1,0)\}$.

5.3 Coloring Arguments

Let $\mathbf{T}$ be a set of tiles. A **coloring argument** for $\mathbf{T}$ is a function $f : \mathbb{N}^2 \to \mathbb{C}$ such that

$$f(\kappa) := \sum_{(a,b)\in\kappa} f(a,b) = 0$$

for any $\kappa \subset \mathbb{N}^2$ which is a translate of a tile in $\mathbf{T}$. It is not difficult to check that the set of coloring arguments for $\mathbf{T}$ forms a vector space over $\mathbb{C}$, which we denote $\mathbb{O}(\mathbf{T})$ and shall call the **coloring space**.

If $R \subset \mathbb{N}^2$ is some region, then we say that a coloring argument $f \in \mathbb{O}(\mathbf{T})$ **forbids** R if $f(R) \neq 0$. If a coloring argument f forbids R then one immediately deduces that R is not tileable by $\mathbf{T}$. If we replace black and white by $+1$ and -1, then the chessboard coloring gives the following coloring argument

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-1	$+1$	-1	$+1$	$\cdots$
$+1$	-1	$+1$	-1	$\cdots$
-1	$+1$	-1	$+1$	$\cdots$
$+1$	-1	$+1$	-1	$\cdots$

(which has formula $f(a,b) = (-1)^{a+b}$) for the tile set $\mathbf{T} = \{V, H\}$ consisting of the two dominoes.

5.4 Tile Polynomials

Let us consider the polynomial ring $\mathbb{C}[x,y]$ in two variables, where $\mathbb{C}$ denotes the complex numbers. To each box $(a,b) \in \mathbb{N}^2$ in the first quadrant we associate the monomial $x^a y^b$:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
y^3	xy^3	x^2y^3	x^3y^3	$\cdots$
y^2	xy^2	x^2y^2	x^3y^2	$\cdots$
y	xy	x^2y	x^3y	$\cdots$
1	x	x^2	x^3	$\cdots$

To each region R (or tile τ) we associate the region (or tile) polynomial

$$p_R(x,y) = \sum_{(a,b)\in R} x^a y^b \in \mathbb{C}[x,y].$$

Thus, $p_V(x, y) = 1 + y$ and $p_H(x, y) = 1 + x$.

We note that translating a tile τ in the direction (a, b) corresponds to multiplying the tile polynomial by $x^a y^b$. Our assumption that the tiles $\tau \in \mathbf{T}$ are southwest-justified means that each $p_\tau(x, y)$ is not divisible by a monomial.[2]

When is a region R tileable by $\mathbf{T}$? This happens exactly when

$$p_R(x, y) = \sum_{(a,b),\tau} x^a y^b p_\tau(x, y), \tag{5.1}$$

where the summation is over some collection of translated tiles.

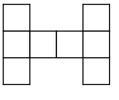
5.5 Tile Ideal

Let us define the **tile ideal** $I_\mathbf{T} \subset \mathbb{C}[x, y]$ to be the ideal generated by the tile polynomials p_τ as τ varies over the tiles in $\mathbf{T}$. A typical element of $p(x, y) \in I_\mathbf{T}$ is thus a finite linear combination

$$p(x, y) = q_1(x, y)p_{\tau_1}(x, y) + \cdots + q_k(x, y)p_{\tau_k}(x, y), \tag{5.2}$$

where $\tau_i \in \mathbf{T}$ are tiles and $q_i(x, y) \in \mathbb{C}[x, y]$. In particular, if a region R is tileable by $\mathbf{T}$ then looking at (5.1) we see that $p_R \in I_\mathbf{T}$. However, the converse is not true. The polynomials $q_i(x, y)$ in (5.2) may involve negative signs which would allow one to "remove" tiles. Let us say that a region R is **tileable by T over** $\mathbb{C}$ if $p_R \in I_\mathbf{T}$. Tileability over $\mathbb{C}$ is a much easier problem, as we shall soon see.

For example, letting $R = \{(0, 0), (0, 1), (0, 2), (1, 1), (2, 1), (3, 0), (3, 1), (3, 2)\}$, we obtain:



It is easy to see that R is not tileable by the dominoes $\mathbf{T} = \{V, H\}$. However we have

$$\begin{aligned} p_R(x, y) &= 1 + y + y^2 + xy + x^2y + x^3 + x^3y + x^3y^2 \\ &= (1 + y + y^2 + x^2 + x^2y + x^2y^2 - x - xy^2)p_H(x, y) \in I_\mathbf{T}, \end{aligned}$$

so R is tileable by dominoes over $\mathbb{C}$.

5.6 Reduction to Finite Sets of Tiles

A basic theorem in commutative algebra is the **Hilbert Basis Theorem**. In our setting, it states that

Theorem 1 (Hilbert Basis Theorem)**.** *Every ideal I in a polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$ is finitely generated. Furthermore, if $S \subset I$ is any possibly infinite set of generators, then a finite subset $S' \subset S$ will generate I.*

Corollary 2. *Any possibly infinite set $\mathbf{T}$ of tiles can be replaced by a finite subset $\mathbf{T}' \subset \mathbf{T}$ of tiles, so that tileability by $\mathbf{T}$ over $\mathbb{C}$ is the same as tileability by $\mathbf{T}'$ over $\mathbb{C}$.*

Proof. Apply Theorem 1 to the tile ideal $I_\mathbf{T} \subset \mathbb{C}[x, y]$. □

5.7 Tiling Over $\mathbb{C}$ and Coloring Arguments

Proposition 3. *We have an isomorphism of $\mathbb{C}$-vector spaces*

$$\mathbb{O}(\mathbf{T}) \simeq \mathrm{Hom}_\mathbb{C}(\mathbb{C}[x, y]/I_\mathbf{T}, \mathbb{C}).$$

[2] We can avoid having to make these assumptions by using the ring $\mathbb{C}[x, y, x^{-1}, y^{-1}]$ instead, but that makes other things somewhat more complicated.

Proof. Let $f \in \mathbb{O}(\mathbf{T})$. We define a $\mathbb{C}$-linear map $\phi : \mathbb{C}[x,y] \to \mathbb{C}$ by the formula

$$\phi(x^a y^b) = f(a,b)$$

and extending by linearity. Since f is a coloring argument, the map ϕ descends to a well-defined map $\bar{\phi} : \mathbb{C}[x,y]/I_{\mathbf{T}} \to \mathbb{C}$. This defines a $\mathbb{C}$-linear map $\mathbb{O}(\mathbf{T}) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$.

In the other direction, let $\bar{\phi} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$. We define $f : \mathbb{N}^2 \to \mathbb{C}$ by the formula

$$f(a,b) = \bar{\phi}(x^a y^b \bmod I_{\mathbf{T}}).$$

This f lies in $\mathbb{O}(\mathbf{T})$ and the resulting map $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C}) \to \mathbb{O}(\mathbf{T})$ is inverse to the one in the previous paragraph. □

It is now time for one of the main results in this article.

Theorem 4. *A region $R \subset \mathbb{N}^2$ is tileable by $\mathbf{T}$ over $\mathbb{C}$ if and only if no coloring argument $f \in \mathbb{O}(\mathbf{T})$ forbids R.*

Proof. The "only if" statement is obvious. To prove the "if" direction, we suppose that R is not tileable by $\mathbf{T}$ over $\mathbb{C}$ so that $p_R(x,y) \notin I_{\mathbf{T}}$. But this means the image $\bar{p}_R(x,y) \in \mathbb{C}[x,y]/I_{\mathbf{T}}$ is a non-zero vector in the $\mathbb{C}$-vector space $\mathbb{C}[x,y]/I_{\mathbf{T}}$. There is thus a map $\bar{\phi} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$ such that $\bar{\phi}(\bar{p}_R) \neq 0$. Using the isomorphism of Proposition 3 this gives a coloring argument $f \in \mathbb{O}(\mathbf{T})$ such that $f(R) \neq 0$. □

5.8 Nullstellensatz and Varieties

Let $I \subset \mathbb{C}[x,y]$ be an ideal. We define the **variety** $V(I)$ of I to be

$$V(I) = \{(\alpha,\beta) \in \mathbb{C}^2 \mid p(\alpha,\beta) = 0 \ \text{ for every } p(x,y) \in I\}.$$

If $X \subset \mathbb{C}^2$ is a set of points in the plane we define the **ideal** $I(X) \subset \mathbb{C}[x,y]$ **of** X by

$$I(X) = \{p(x,y) \in \mathbb{C}[x,y] \mid p(\alpha,\beta) = 0 \ \text{ for every } (\alpha,\beta) \in X\}.$$

(One can obviously make these definitions in dimensions more than two.)

An ideal I in a commutative ring B is called **radical** if for any $b \in B$ such that $b^n \in I$ we have $b \in I$. For example, the ideal $\langle 1+x, 1+y \rangle \subset \mathbb{C}[x,y]$ that we have previously seen is radical. A fundamental result in commutative algebra and algebraic geometry is **Hilbert's Nullstellensatz**.

Theorem 5 (Nullstellensatz)**.** *Let $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n]$ be an ideal not equal to the whole polynomial ring. Then $V(I)$ is non-empty. Furthermore, if I is radical then we have $I(V(I)) = I$.*

5.9 Tile Variety

Theorem 4 is satisfying theoretically, but to solve our favorite tiling problems it would be nice to exhibit an explicit basis for $\mathbb{O}(\mathbf{T})$. By Proposition 3, the dimension of $\mathbb{O}(\mathbf{T})$ is equal to that of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$. If $\mathbb{C}[x,y]/I_{\mathbf{T}}$ is infinite-dimensional over $\mathbb{C}$ (it will always be of countable dimension), then $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$ will be of uncountable dimension. As an example, take $\mathbf{T} = \{V\}$ to consist of only the vertical domino. Then $\mathbb{C}[x,y]/I_{\mathbf{T}} \simeq \mathbb{C}[x]$ is infinite-dimensional over $\mathbb{C}$. For simplicity we will assume that $\mathbb{C}[x,y]/I_{\mathbf{T}}$ and thus $\mathbb{O}(\mathbf{T})$ is a finite-dimensional $\mathbb{C}$-vector space.[3]

Define the tile variety $V_{\mathbf{T}} = V(I_{\mathbf{T}}) \subset \mathbb{C}^2$ to be the variety associated to the ideal $I_{\mathbf{T}}$. For example, if $\mathbf{T} = \{V, H\}$ then $V_{\mathbf{T}}$ is given by the set of common zeroes of $1+x$ and $1+y$. Thus $V_{\mathbf{T}} = \{(-1,-1)\}$. It will follow from Theorem 6 below that if $\mathbb{C}[x,y]/I_{\mathbf{T}}$ is finite-dimensional over $\mathbb{C}$ then $V_{\mathbf{T}}$ is a finite set of points.

[3] The description we now give will not lead to a basis for $\mathbb{O}(\mathbf{T})$ in the infinite-dimensional case, but other techniques such as **Gröbner bases** can still tackle the general case.

For a point $(\alpha, \beta) \in V_{\mathbf{T}}$ define a map $\bar{\phi}_{\alpha,\beta} \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$ by evaluating polynomials at (α, β):

$$\bar{\phi}_{\alpha,\beta}(p(x,y)) = p(\alpha, \beta).$$

Note that this equations is well-defined exactly because $(\alpha, \beta) \in V_{\mathbf{T}}$. These elements of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$ are very special: they are not just linear maps, but also $\mathbb{C}$-algebra homomorphisms of $\mathbb{C}[x,y]/I_{\mathbf{T}}$ to $\mathbb{C}$. Under the isomorphism of Proposition 3, $\bar{\phi}_{\alpha,\beta}$ corresponds to the coloring argument $f : \mathbb{N}^2 \to \mathbb{C}$ given by $f_{\alpha,\beta}(a,b) = \alpha^a \beta^b$.

Perhaps you now see where we are heading. If we take $\mathbf{T} = \{V, H\}$ to consist of the two dominoes, and $(\alpha, \beta) = (-1,-1)$ then $f_{-1,-1}(a,b) = (-1)^{a+b}$ is just the black-white chessboard coloring!

5.10 A Basis for the Coloring Space

Theorem 6. *Suppose $\mathbb{C}[x,y]/I_{\mathbf{T}}$ has dimension n over $\mathbb{C}$ and $I_{\mathbf{T}}$ is a radical ideal. Then $V_{\mathbf{T}} = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ consists of n points and the set $\{f_{\alpha_i,\beta_i} \in \mathbb{O}(\mathbf{T})\}$ forms a basis of the coloring space $\mathbb{O}(\mathbf{T})$.*

Proof. We claim that an element $\bar{p}(x,y) \in \mathbb{C}[x,y]/I_{\mathbf{T}}$ is completely determined by its values $\bar{p}(\alpha_i, \beta_i)$ on $V_{\mathbf{T}}$. This follows from Theorem 5: if $p, q \in \mathbb{C}[x,y]$ take the same values everywhere on $V_{\mathbf{T}}$ then the difference $p - q$ lies in $I(V_{\mathbf{T}})$ and thus in $I_{\mathbf{T}}$ by the Nullstellensatz. In particular, we have

$$\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}) \leq |V_{\mathbf{T}}|.$$

But if $\{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\} \subset V_{\mathbf{T}}$ and $j \in [1,m]$ is fixed let us pick for each $i \neq j$ in $[1,m]$ a polynomial

$$q_i^{(j)}(x,y) = \frac{x - \alpha_i}{\alpha_j - \alpha_i} \quad \text{or} \quad q_i^{(j)}(x,y) = \frac{y - \beta_i}{\beta_j - \beta_i},$$

insisting that we choose an expression such that the denominator is non-zero (most of the time either one will do). Then the product

$$q^{(j)}(x,y) = \prod_{i \neq j} q_i^{(j)}(x,y) \in \mathbb{C}[x,y]$$

takes the value 1 at (α_j, β_j) and the value 0 at every other (α_i, β_i). These m polynomials give m linearly independent elements of $\mathbb{C}[x,y]/I_{\mathbf{T}}$. Thus,

$$\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}) \geq |V_{\mathbf{T}}|$$

and we conclude that $n = \dim_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}) = |V_{\mathbf{T}}|$. In particular, we have shown that $V_{\mathbf{T}} = \{(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)\}$ is finite. One checks that the maps $\{\bar{\phi}_{\alpha_i,\beta_i}\} \subset \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[x,y]/I_{\mathbf{T}}, \mathbb{C})$ form a dual-basis to $\{q^{(j)}(x,y)\} \subset \mathbb{C}[x,y]/I_{\mathbf{T}}$, to complete the proof. □

For $\mathbf{T} = \{V, H\}$, we have remarked that $I_{\mathbf{T}}$ is radical so Theorem 6 says that the chessboard coloring is essentially the only coloring argument. There is also a version of Theorem 6 which applies even when $I_{\mathbf{T}}$ is not radical.

5.11 Summary of Strategy

Let us summarize our approach to a tiling problem. We are given a set $\mathbf{T}$ of tiles and a region R. First, we convert each tile $\tau \in \mathbf{T}$ into a polynomial $p_\tau(x,y)$. We (try to) solve all these polynomials simultaneously, to find the tile variety $V_{\mathbf{T}} \subset \mathbb{C}^2$. If $V_{\mathbf{T}} = \varnothing$ then every region R is tileable by $\mathbf{T}$ over $\mathbb{C}$.

We suppose $V_{\mathbf{T}}$ consists of a finite set of points. Next we evaluate $p_R(x,y)$ at each point (α, β) of $V_{\mathbf{T}}$. If for some point we have $p_R(\alpha, \beta) \neq 0$ then we have found a coloring argument $f_{\alpha,\beta}$ which forbids R. If not, but in addition we know that $I_{\mathbf{T}}$ is radical, then we can conclude from Theorem 6 that no coloring argument can show that R is not tileable. Of course, to completely resolve whether R is tileable by $\mathbf{T}$ is a much harder problem.

Furthermore, all the results so far work in any number of dimensions.

5.12 Final Comments

Essentially all of what we have presented so far is a simplification of work of Barnes [B1, B2]. However, much more can be said if we are willing to restrict our class of tiling problems. Let us now assume that all the tiles and regions that we consider are **bricks**. In two-dimensions, bricks are just rectangles. In d-dimensions, they are regions of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$.

A fundamental result is an analogue of the Hilbert Basis Theorem over $\mathbb{N}$, due to de Bruijn and Klarner.

Theorem 7 ([dBK]). *When considering tiling problems of bricks by bricks, any collection of brick tiles can be replaced by a finite subcollection.*

For brick tiling problems, tiling over $\mathbb{C}$ and usual tilings are not too different. Barnes proved:

Theorem 8 ([B2]). *Let $\mathbf{T}$ be a finite set of brick tiles. Then there is some constant K such that every brick region R with all dimensions greater than K can be tiled by $\mathbf{T}$ if and only if it can be tiled by $\mathbf{T}$ over $\mathbb{C}$.*

Together with Ezra Miller and Igor Pak, I have been studying some computational issues for tilings. I now describe some of our results. Let us say that a set $\mathbf{S}$ of bricks has a **finite description** if it is a finite union $\mathbf{S} = \cup_i \mathbf{S}_i$ of brick classes $\mathbf{S}_i$ such that each class is of one of the following forms:

1. $\{(l_1, \ldots, l_d) \mid l_i = a\}$
2. $\{(l_1, \ldots, l_d) \mid l_i > a\}$
3. $\{(l_1, \ldots, l_d) \mid l_i > a$ and $l_i \equiv b \bmod c$

for integers a, b and c.

Proposition 9 ([LMP]). *Let $\mathbf{T}$ be a set of bricks. Then the set $\mathbf{S}$ of bricks which can be tiled by $\mathbf{T}$ admits a finite description.*

Theorem 10 ([LMP]). *Suppose we are in $d = 2$ dimensions and $\mathbf{T}$ is a finite set of bricks. Then it is possible to compute a finite description for the set $\mathbf{S}$ of bricks tileable by $\mathbf{T}$.*

Surprisingly, we conjecture that Theorem 10 fails in higher dimensions. That is, when $d \geq 3$, a finite description for the set $\mathbf{S}$ of bricks tileable by $\mathbf{T}$ is not computable.

References

[B1] Frank W. Barnes: Algebraic Theory of Brick Packing I, *Discrete Math.* **42** (1982), 7–26.

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[LMP] Thomas Lam, Ezra Miller, and Igor Pak: Brick tilings, in preparation.