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# Hunting for Perfect Euler Bricks

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An **Euler brick** is a cuboid with integer side dimensions $a, b, c$ such that the face diagonals are integers. The cuboid with dimensions $(a, b, c) = (44, 117, 240)$, for example, is an Euler brick. It is the smallest Euler brick. If $(a, b, c)$ is an Euler brick, then $(ka, kb, kc)$ is an Euler brick too for positive integers $k$. If also the space diagonal is an integer, an Euler brick is called a **perfect Euler brick**. In other words, a perfect Euler brick has the properties that all vertex coordinates and vertex distances are integers.

Whether a perfect Euler brick exists is an open mathematical problem. One would have to find integer vectors $(a, b, c)$ such that

$$\sqrt{a^2+b^2}, \sqrt{a^2+c^2}, \sqrt{b^2+c^2}, \sqrt{a^2+b^2+c^2}$$

are integers. Nobody has found a solution to this system of Diophantine equations nor shown that solutions do not exist. A infinite subclass of Euler bricks can be parametrized: if $u, v, w$ is a Pythagorean triple $u^2 + v^2 = w^2$, then

$$(a, b, c) = (|u(4v^2 - w^2)|, |v(4u^2 - w^2)|, |4uvw|)$$

is an Euler brick.

Because $a^2 + b^2 + c^2 = f(t, s)(s^2 + t^2)^2$ if $u = 2st; v = s^2 - t^2; w = s^2 + t^2$ and $f(t, s) = s^8 + 68s^6t^2 - 122s^4t^4 + 68s^2t^6 + t^8$, it would suffice to find $s, t$ for which $f(t, s)$ is a square in order to find a perfect Euler brick. There are many Euler bricks which do not fall into the above Saunderson parametrization known since 1740. A brute force search $1 \leq a \leq b \leq c \leq 8000$ leads to 120 Euler bricks. Only 16 of the 120 Euler bricks in $1 \leq a \leq b \leq c \leq 8000$ are prime bricks, triples $(a, b, c)$ which are not a multiple of a smaller brick. Some of them, like $(85, 132, 720)$ are not of the above parametrization. To look for perfect Euler bricks of the parametrized type we can search for integers $\sqrt{f(t, s)}$ with the help of a computer. Since perfect Euler bricks might not exist, one can try to find Euler bricks $(a, b, c)$ for which $\sqrt{a^2 + b^2 + c^2}$ is as close to an integer as possible. One approach is to linearize the map $T : \sqrt{f(t, s)} \to \sqrt{f(t + u, s + v)} \bmod 1$ for suitable $(u, v)$ and use a continued fraction expansion of the irrational rotation $dT(x) = x + \alpha \bmod 1$ on $[0, 1)$ to find $n$ for which $dT^n(x)$ is close to 0. There exists for example a number $a$ with 68162 digits, a number $b$ with 56802 digits and a number $c$ with 56803 digits so that the diagonal length $\sqrt{a^2 + b^2 + c^2}$ is $10^{-60589}$ close to an integer. Computations with such large numbers push the boundaries of computer algebra systems. One has to take square roots of integers with hundreds of thousands of digits. It turns out that in such ranges, some computer algebra systems have limitations when projecting algebraic numbers to real valued numbers. While the quest for Euler bricks might appear an entertainment without applications, the treasure hunt can at least help to explore the boundaries and limitations of computer algebra systems.

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[†] Oliver Knill got his mathematics degrees at ETH Zuerich in Switzerland and has been in California, Arizona, and Texas before coming to Harvard in 2000. He especially likes to think about mathematical topics which are accessible to undergraduates and which also allow exploration, visualization, and experimentation with the help of computers.