

# An Unconventional Inequality

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## 10.1 Introduction

Problem solving has been an important aspect of mathematics in my life. It is the challenge of tackling a math problem and experiencing the moments of both insight and perplexity associated with the problem solving process that have drawn me to the field of mathematics.

When asked to write about my “favorite” problem, I found it rather difficult to single out a particular problem from my repository of interesting mathematics problems. Though I do not have a single favorite problem, I tried to choose a problem which meets several criteria that make a mathematics problem interesting. First, the problem should have a simple/elegant solution; at the same time, the solution should have some key step which is clever and difficult. The ideal problem is the one that looks intractable upon first sight but, after one has read the solution, should evoke the response, “Ah, that was simple! Why didn’t I think of that?” Moreover, the problem should have interesting generalizations or connections to other problems or ideas in mathematics.

Keeping in mind these characteristics of a good problem, I have selected one which I believe meets the criteria. But first, I will present some background on the problem. The problem was created by Reid Barton and submitted as a problem for the 2003 International Mathematical Olympiad (IMO). Though it was not selected as a question on the IMO, it was included on the IMO Shortlist, an annual list of twenty to thirty problems which are in contention for a place on the IMO exam. The problem was first presented to me during the 2004 USA Mathematical Olympiad Summer Program (MOSP), where it was given as a problem on a practice test. As I recall, no one was able to solve it, and I was quite fascinated after learning its clever solution.

## 10.2 Problem

The problem is stated below:

**Problem.** Let  $n$  be a positive integer and let  $(x_1, \dots, x_n)$ ,  $(y_1, \dots, y_n)$  be two sequences of positive real numbers. Suppose  $(z_2, \dots, z_{2n})$  is a sequence of positive real numbers such that

$$z_{i+j}^2 \geq x_i y_j \text{ for all } 1 \leq i, j \leq n. \quad (10.1)$$

Let  $M = \max\{z_2, \dots, z_{2n}\}$ . Prove that

$$\left( \frac{M + z_2 + \dots + z_{2n}}{2n} \right)^2 \geq \left( \frac{x_1 + \dots + x_n}{n} \right) \left( \frac{y_1 + \dots + y_n}{n} \right). \quad (10.2)$$

The first thing that strikes the reader about this problem is how unconventional it is. The condition given in (10.1) is certainly bizarre, as is the appearance of  $M$  in the inequality. Proving

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inequalities is a common type of question in the olympiad exams, and any experienced olympiad problem solver has in his arsenal a number of tools to attack such questions, such as AM-GM inequality, Cauchy-Schwarz inequality, and Muirhead’s inequality (see [HLP]), to name a few. The trouble with this inequality is that none of the standard tricks seem to work, as we will highlight.

The first questions that emerge regarding the inequality are how strict it is and what equality cases, if any, there are. Upon quick inspection, one notices that choosing  $x_1 = x_2 = \dots = x_n = y_1 = y_2 = \dots = y_n = z_2 = z_3 = \dots = z_{2n}$  satisfies (10.1) and yields equality in our inequality. This equality condition, combined with the nature of the left-hand side of (10.2), is reminiscent of the Inequality of Arithmetic and Geometric Means (or AM-GM, for short):

**Theorem 1** (AM-GM). *Given a list of  $k$  nonnegative real numbers  $a_1, a_2, \dots, a_k$ , the following inequality holds:*

$$\frac{a_1 + a_2 + \dots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \dots a_k},$$

with equality if and only if  $a_1 = a_2 = \dots = a_k$ .

A simple proof of this inequality can be found in ([HLP]).

Since the equality case of (10.2) appears to require  $M = z_2 = z_3 = \dots = z_n$ , we are tempted to try applying the AM-GM inequality on the left-hand side of (10.2):

$$\frac{M + z_2 + z_3 + \dots + z_{2n}}{2n} \geq \sqrt[2n]{M z_2 z_3 \dots z_{2n}}.$$

Thus, proving (10.2) reduces to showing that

$$\sqrt[2n]{M z_2 z_3 \dots z_{2n}} \geq \left( \frac{x_1 + \dots + x_n}{n} \right) \left( \frac{y_1 + \dots + y_n}{n} \right).$$

However, the problem is that the above inequality is actually false, in general. One can come up with numerous counterexamples; for instance, take  $n = 3$  with  $x_1 = y_1 = 1, x_2 = y_2 = 2, x_3 = y_3 = 3, z_2 = 1, z_3 = \sqrt{2}, z_4 = 2, z_5 = \sqrt{6}$ , and  $z_6 = 3$ , and the above inequality is not satisfied. Thus, this approach does not work, as AM-GM is too weak for the left-hand side.

Another approach is to use the Cauchy-Schwarz inequality, which in sequence form, states the following:

**Theorem 2** (Cauchy-Schwarz Inequality). *Given  $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$ , we have*

$$\left( \sum_{i=1}^n x_i^2 \right) \left( \sum_{i=1}^n y_i^2 \right) \geq \left( \sum_{i=1}^n x_i y_i \right)^2.$$

The trouble is that the only apparent candidate for application of Cauchy-Schwarz would be the right side of (10.2), in the form of

$$\left( \frac{x_1 + \dots + x_n}{n} \right) \left( \frac{y_1 + \dots + y_n}{n} \right) \geq \left( \frac{\sqrt{x_1 y_1}}{n} + \dots + \frac{\sqrt{x_n y_n}}{n} \right)^2.$$

However, this inequality goes in the wrong direction, so we are forced to abandon this idea.

Another idea uses a different application of AM-GM, this time on the right side of (10.2). The right side is a product of two quantities and lends itself to AM-GM:

$$\begin{aligned} \left( \frac{x_1 + \dots + x_n}{n} \right) \left( \frac{y_1 + \dots + y_n}{n} \right) &\leq \left( \frac{\frac{x_1 + \dots + x_n}{n} + \frac{y_1 + \dots + y_n}{n}}{2} \right)^2 \\ &= \left( \frac{x_1 + \dots + x_n + y_1 + \dots + y_n}{2n} \right)^2. \end{aligned}$$

In light of the above inequality, it suffices to establish

$$M + z_2 + z_3 + \cdots + z_{2n} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n,$$

which seems to be a simpler inequality. The reader can try some examples and convince himself that the above inequality appears to be true. Thus, this line of attack seems promising.

However, in attempting to prove (10.2), none of the standard tricks appear to work. The tricky part lies in using the condition (10.1) effectively. One could reasonably expect to use  $z_k \geq \sqrt{x_i y_j}$  for some  $i + j$  and prove some inequality of the form

$$M + \sum \sqrt{x_i y_j} \geq x_1 + \cdots + x_n + y_1 + \cdots + y_n,$$

where the sum ranges over certain pairs  $(i, j)$ . However, the direction of the inequality makes it nearly intractable to tackle via inequalities such as AM-GM or Muirhead.

Justifiably so, it is at this juncture that a brilliant maneuver is required. First, we take advantage of the homogeneity of (10.1) and (10.2). Let  $x = \max\{x_1, \dots, x_n\}$  and  $y = \max\{y_1, \dots, y_n\}$ . Then without loss of generality, one may replace each  $x_i$  with  $x'_i = x_i/x$ , each  $y_i$  with  $y'_i = y_i/y$ , and each  $z_i$  with  $z'_i = z_i/\sqrt{xy}$  without affecting the statement of the problem. Thus, it suffices to prove (10.2) under the added assumption that  $\max\{x_1, x_2, \dots, x_n\} = \max\{y_1, y_2, \dots, y_n\} = 1$ .

Now, the critical ingredient in the proof is the following lemma:

**Lemma 3.** *Let  $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$  be positive reals. Suppose that, for any  $r > 0$ , the following property is satisfied:*

- (i). *The number of  $i$  for which  $a_i > r$  is at least the number of  $i$  for which  $b_i > r$ .*

*Then,  $a_1 + \cdots + a_n \geq b_1 + \cdots + b_n$ .*

*Proof.* Without loss of generality, one can assume that  $a_1 \leq a_2 \leq \cdots \leq a_k$  and  $b_1 \leq b_2 \leq \cdots \leq b_k$ . Note that if there exists  $k$  for which  $b_k > a_k$ , then, the number of  $i$  for which  $b_i > (a_k + b_k)/2$  is at least  $n - k + 1$  (since  $i = k + 1, \dots, n$  satisfy the relation), while the number of  $i$  for which  $a_i > (a_k + b_k)/2$  is at most  $n - k$ , contradicting the initial assumption. Hence, we must have  $a_i \geq b_i$  for all  $i$ , and so,  $a_1 + \cdots + a_n \geq b_1 + \cdots + b_n$ .  $\square$

The lemma seems rather obvious, but it is powerful enough to establish (10.2) and thus provide a complete solution.

## 10.3 Solution

We now present a complete solution (provided by [DJMP]) to the problem along these lines:

*Proof.* Let  $x = \max\{x_1, \dots, x_n\}$  and  $y = \max\{y_1, \dots, y_n\}$ . Then, without loss of generality, we may assume that  $x = y = 1$ , for we can always replace  $x_i$  by  $x_i/x$ ,  $y_i$  by  $y_i/y$ , and  $z_i$  by  $z_i/\sqrt{xy}$  without affecting the statement of the problem. It suffices to show that

$$M + z_2 + \cdots + z_{2n} \geq (x_1 + \cdots + x_n) + (y_1 + \cdots + y_n)$$

because applying the AM-GM inequality to the sum of two terms on the right side of the above inequality would give us the desired result.

Now, by the previous lemma, we need only show that, for any  $r > 0$ , the number of terms on the left side of (10.2) that are greater than  $r$  is at least the number of those on the right side. Observe that if  $r \geq 1$ , this property clearly holds, as no terms on the right side are greater than  $r$ .

Next, suppose  $r < 1$ . Let  $X = \{i : x_i > r\}$ ,  $Y = \{i : y_i > r\}$ , and  $Z = \{i : z_i > r\}$ . Note that if  $x_i, y_j > r$ , then  $z_{i+j} \geq \sqrt{x_i y_j} > r$ . Thus,

$$\{i + j : i \in X, j \in Y\} \subseteq Z.$$

However, note that  $X$  and  $Y$  are nonempty (since  $r < 1$  and  $x = y = 1$ ). Thus, if  $X = \{a_1, a_2, \dots, a_l\}$  and  $Y = \{b_1, b_2, \dots, b_m\}$  with  $a_1 < a_2 < \cdots < a_l$  and  $b_1 < b_2 < \cdots < b_m$ ,

then  $\{a_1 + b_1, a_1 + b_2, \dots, a_1 + b_m, a_2 + b_m, a_3 + b_m, \dots, a_m + b_m\} \subseteq Z$ , which shows that  $|Z| \geq |X| + |Y| - 1$ . But then, we also have that  $M > r$ . Hence, there are at least  $|X| + |Y|$  elements on the left side of (10.2) that are greater than  $r$ .

This concludes the proof. □

The proof is everything we want it to be: short, elegant, and clever.

### 10.4 Further Connections

What I find remarkable about the problem, apart from the simple (albeit difficult) nature of its solution, is the multitude of connections it has with the field of discrete geometry. As it turns out, the inequality we have discussed is related to an important inequality known as the Prékopa-Leindler Inequality:

**Theorem 4** (Prékopa-Leindler Inequality). *Let  $0 < \lambda < 1$ , and let  $f, g, h : \mathbb{R}^n \rightarrow [0, \infty)$  be measurable functions such that*

$$h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$$

for all  $x, y \in \mathbb{R}^n$ . Then,

$$\int_{\mathbb{R}^n} h(x) dx \geq \left( \int_{\mathbb{R}^n} f(x) dx \right)^\lambda \left( \int_{\mathbb{R}^n} g(x) dx \right)^{1-\lambda}.$$

Of course, the most striking difference between the given problem and the statement of the Prékopa-Leindler Inequality is the fact that the former deals with sequences while the latter deals with functions. In fact, our inequality can be viewed as a discrete specialization of the Prékopa-Leindler inequality for  $n = 1$  and  $\lambda = 1/2$ .

The Prékopa-Leindler inequality has many important applications, such as probability theory, optimal mass transportation [Vi], and the theory of diffusion [BL]. Perhaps its most important consequence is the Brunn-Minkowski inequality (see [Ba]), which can be stated as follows (note that there exist alternative formulations of the inequality):

**Theorem 5.** *If  $A$  and  $B$  are compact subsets of  $\mathbb{R}^n$ , then*

$$|\lambda A + (1 - \lambda)B|^{1/n} \geq \lambda|A|^{1/n} + (1 - \lambda)|B|^{1/n},$$

where  $|X|$  denotes the Lebesgue measure of  $X$ , and  $\lambda A + (1 - \lambda)B$  denotes the Minkowski sum  $\{\lambda a + (1 - \lambda)b : a \in A, b \in B\}$ .

The Brunn-Minkowski inequality can be used to provide a simple proof (from [Ba]) of the famous isoperimetric inequality:

**Theorem 6** (Isoperimetric Inequality). *Among simple closed bodies of a given volume in  $\mathbb{R}^n$ , Euclidean balls have the least surface area.*

*Proof.* Let  $C \in \mathbb{R}^n$  be a compact set with volume equal to that of  $B_n$ , the Euclidean ball of radius 1. Then, the surface area of  $C$  is given by

$$|\partial C| = \lim_{\epsilon \rightarrow 0} \frac{|C + \epsilon B_n| - |C|}{\epsilon}.$$

By the Brunn-Minkowski inequality, we have

$$|C + \epsilon B_n| \geq \left( |C|^{1/n} + \epsilon |B_n|^{1/n} \right)^n \geq |C| + n\epsilon |C|^{(n-1)/n} |B_n|^{1/n}.$$

It then follows that

$$\begin{aligned} |\partial C| &\geq n|C|^{(n-1)/n} |B_n|^{1/n} \\ &= n|B_n|. \end{aligned}$$

Using the well-known fact  $n|B_n| = |\partial B_n|$  (see page 4 of [Ba]), we obtain  $|\partial C| \geq |\partial B_n|$ , as desired. □

## References

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