FEATURE

11 Problems

The HCMR welcomes submissions of original problems in any fields of mathematics, as well as solutions to previously proposed problems. Proposers should direct problems to `hcmr-problems@hcs.harvard.edu` or to the address on the inside front cover. A complete solution or a detailed sketch of the solution should be included, if known. Solutions to previous problems should be directed to `hcmr-solutions@hcs.harvard.edu` or to the address on the inside front cover. Solutions should include the problem reference number, as well as the solver's name, contact information, and affiliated institution. Additional information, such as generalizations or relevant bibliographical references, is also welcome. Correct solutions will be acknowledged in future issues, and the most outstanding solutions received will be published. To be considered for publication, solutions to the problems below should be postmarked no later than *April 13, 2009*; any problems not solved in this admittedly short window will be reopened in Vol. 3, No. 1 with a solution submission deadline of September 21, 2009. An asterisk beside a problem or part of a problem indicates that no solution is currently available.

F08 – 1. Let p, q be two positive integers, and let n be integers such that $n \geq p+q$. Prove that the following identity holds:

$$\sum_{i=0}^{p} \binom{p}{i}\binom{q}{p-i}\binom{n+i}{p+q} = \sum_{i=0}^{p} \binom{p}{i}\binom{n}{i}\binom{n-i}{q}.$$

Proposed by Cosmin Pohoata (Bucharest, Romania).

F08 – 2. Let p be an odd prime. For every positive integer n, let

$$A(n) = 1^n + 2^n + \cdots + (p-2)^n \quad \text{and} \quad B(n) = 1^n + (p-1)^n.$$

Let $\{a_i\}_{i=1}^{\infty}$ be the sequence defined by $a_1 = 2, a_2 = p^2 + 2$ and

$$\begin{cases} a_{n+2} = A(n)a_{n+1} + B(n)a_n & \text{if } p \nmid n+1, \\ a_{n+2} = [A(n) + B(n)]a_{n+1} + a_n & \text{if } p \mid n+1. \end{cases}$$

Prove that no a_n is equal to the product of any $p-1$ terms of the sequence $\{a_i\}_{i=1}^{\infty}$.

Proposed by Daniel Campos Salas (Costa Rica).

F08 – 3. Let $f : [0,1] \to \mathbb{R}$ be a differentiable function with continuous derivative such that

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx.$$

Prove that there exists $\xi \in (0,1)$ such that

$$f(\xi) = f'(\xi)\int_0^{\xi} f(x)\,dx.$$

Proposed by Cezar Lupu (University of Bucharest, Bucharest, Romania).

F08 – 4. Do there exist functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

- both are periodic, *i.e.* there exist nonzero real a, b such that for all $x \in \mathbb{R}$, $f(x) = f(x+a)$ and $g(x) = g(x+b)$, and
- their sum is equal to the identity, *i.e.* for all $x \in \mathbb{R}$, $f(x) + g(x) = x$?

Proposed by Robert Obryk (August Witkowski High School, Krakow, Poland).

F08 – 5. Let ABC be an arbitrary triangle and let I be the incenter of ABC. Let D, E, F be the points on lines $\overline{BC}, \overline{CA}, \overline{AB}$ respectively such that $\angle BID = \angle CIE = \angle AIF = 90^\circ$, and define the following measurements: r_a, r_b, r_c are the exradii of the triangle ABC, Δ' is the area of DEF, and Δ is the area of ABC. Prove that

$$\frac{\Delta'}{\Delta} = \frac{4r(r_a + r_b + r_c)}{(a+b+c)^2}.$$

Proposed by Mehmet Şahin (Ankara, Turkey).

The following two problems from the Spring 2008 issue are being released for one more issue. The first required correction and clarification, and we are grateful to Daniel Kane, G2 for bringing these issues to our attention. The second problem below received no solutions.

S08 – 2. Professor Perplex is at it again! This time, he has gathered his $n > 0$ combinatorial electrical engineering students and proposed:

> "I have prepared a collection of $r > 0$ identical *and indistinguishable* rooms, each of which is empty except for $s > 0$ switches *all initially set to the 'off' position*. You will be let into the rooms at random, in such a fashion that no two students occupy the same room at the same time and every student will visit each room arbitrarily many times. Once one of you is inside a room, he or she may toggle any of the s switches before leaving. This process will continue until some student chooses to assert that all the students have visited all the rooms at least $v > 0$ times each. If this student is right, then there will be no final exam this semester. Otherwise, I will assign a week-long final exam on the history of the light switch."

What is the minimal value of s (as a function of n, r, and v) for which the students can guarantee that they will not have to take an exam?

Proposed by Scott D. Kominers '09, Paul Kominers (MIT '12), and Justin Chen (Caltech '09).

S08 – 4. Consider a, b, c three arbitrary positive real numbers. Prove that

$$\sum_{cyc} \sqrt{\frac{b+c}{a}} \geq 2 \left(\sum_{cyc} \sqrt{\frac{a}{b+c}} \right) \cdot \sqrt{1 + \frac{(a+b)(b+c)(c+a) - 8abc}{4 \sum_{cyc} a(a+b)(a+c)}}.$$

Proposed by Cosmin Pohoata (Bucharest, Romania).