FEATURE

12 Solutions

Seeing Stars

F07 – 5. For $i = 1, \ldots, n$, let $f_i : (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n \to (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n$ be given by

$$f_i\left((x_1, \ldots, x_n)\right) = \begin{cases} (\star, x_2 + 1, x_3, \ldots, x_n) & i = 1 \text{ and } x_1 = 1, \\ (x_1, \ldots, x_{i-1} + 1, \star, x_{i+1} + 1, \ldots, x_n) & 1 < i < n \text{ and } x_i = 1, \\ (x_1, \ldots, x_{n-2}, x_{n-1} + 1, \star) & i = n \text{ and } x_n = 1, \\ (x_1, \ldots, x_n) & \text{otherwise,} \end{cases}$$

where $\star + 1 = \star$. Find necessary and sufficient conditions on $(x_1, \ldots, x_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ such that there exists a sequence $\{i_k\}_{k=1}^n$ for which

$$f_{i_n}(\cdots(f_{i_1}((x_1, \ldots, x_n)))) = (\star, \ldots, \star).$$

Proposed by Paul Kominers (Walt Whitman HS '08), Scott D. Kominers '09, and Zachary Abel '10.

Solution by Benjamin Dozier '12. Call an n-tuple $(x_1, \ldots, x_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ **starrable** if and only if there exists a sequence $\{i_k\}_{k=1}^n$ with

$$(f_{i_n} \circ \cdots \circ f_{i_1})(x_1, \ldots, x_n) = (\star, \ldots, \star). \tag{12.1}$$

We call any sequence $\{i_k\}_{k=1}^n$ for which (12.1) holds a **starring sequence for $(x_1, \ldots, x_n)$**.

Call an n-tuple $(x_1, \ldots, x_n)$ **uninull** if and only if $\sum_{j=1}^k x_j = 0$ or 1 for $1 \leq k \leq n-1$ and $\sum_{j=1}^n x_j = 1$. We claim that the starrable n-tuples are precisely those that are uninull.

If $m = 1$ then the only n-tuple is $(0, \ldots, 0)$, which is both starrable and uninull. For the rest of the proof, we assume $m > 1$.

We prove the claim by induction on n. For $n = 1$, the only 1-tuple that is starrable is (1), which is also the only 1-tuple that is uninull. Assume the claim for n. Let $A = (x_1, \ldots, x_{n+1})$ be a starrable $(n+1)$-tuple with starring sequence $\{i_k\}_{k=1}^{n+1}$. First note that

(i) initially none of the elements of A are stars,

(ii) they all become stars after we have applied a starring sequence of functions, and

(iii) if f_j changes the ith element of an n-tuple from a non-star to a star then $i = j$.

We conclude that the set of all elements of the starring sequence equals the set $\{1, 2, \ldots, n\}$. In particular, all of the elements of the starring sequence are distinct.

Now if x_1 is not equal to 0 or 1, then $(f_{i_n} \circ \cdots \circ f_{i_1})(A)$ will have first element not equal to $\star$, since the only functions that can change the first element are f_1 and f_2, but f_2 either increments the first element by 1 or does not affect the first element and f_1 changes the first element from 1 to $\star$. This is a contradiction since A is starrable and $\{i_k\}_{k=1}^{n+1}$ is a starring sequence for A; thus x_1 is equal to either 1 or 0.

If $x_1 = 0$ it is easy to see that 2 must precede 1 in the sequence $\{i_k\}$ since f_1 only outputs a sequence that has a star as the first element if the first element of the input is 1 or $\star$ and f_2 must have already been applied for this to be the case. Since f_1 affects at most two elements of its input $(n+1)$-tuple, the first and the second, and since f_2 is applied before f_1, changing the second element to a star which will remain a star even after f_1 is applied, we conclude that f_1 only affects the first element of the $(n+1)$-tuple. Thus $(x_2, \ldots, x_{n+1})$ must be a starrable n-tuple, and, by the inductive hypothesis, also uninull. But then $(0, x_2, \ldots, x_{n+1})$ is uninull.

Now we consider the case when $x_1 = 1$. It is easy to see that 2 must come after 1 in the sequence $\{i_k\}$ because f_1 and f_2 are the only functions that can affect the first element, but if f_2 is applied before f_1 then the first element becomes $2 \neq 1$, and thus f_1 will have no effect. Thus the effect of f_1 will be to change the first element to a star and increase the second element by 1. It then follows that $(x_2 - 1, x_3, \ldots, x_{n+1})$ must be a starrable n-tuple, and, by the inductive hypothesis, also uninull. But then $(1, x_2, x_3, \ldots, x_{n+1})$ is uninull.

Combining the $x_1 = 0$ and $x_1 = 1$ cases we see that any starrable $n+1$ tuple must also be uninull. Now we prove the converse.

Let $(a_1, \ldots, a_n)$ be a starrable n-tuple in $(\mathbb{Z}/m\mathbb{Z})^n$ with starring sequence $i'_1, \ldots, i'_n$. Then $(0, a_1, \ldots, a_n)$ is a starrable $(n+1)$-tuple with starring sequence $i'_1, \ldots, i'_n, 1$. Also, $(1, a_1 - 1, \ldots, a_n)$ is a starrable $(n+1)$-tuple with starring sequence $1, i'_1, \ldots, i'_n$.

Now note that if an n-tuple $(x_1, \ldots, x_n)$ is uninull then x_1 is either 0 or 1. If the former holds, then $(x_2, \ldots, x_n)$ is also uninull and thus starrable by the inductive hypothesis. But then, as discussed above, $(x_1, \ldots, x_n)$ is starrable. Alternatively, if $x_1 = 1$ then $(x_2 + 1, x_3, \ldots, x_n)$ is uninull—thus starrable by the inductive hypothesis—and $(x_1, \ldots, x_n)$ is again starrable by the previous paragraph. □

Also solved by Kenfin Tomioka (University of Tokyo, Japan) and the proposers.

Symmetrized Sudoku Kernels

S08 – 1. It is known that there are 6670903752021072936960 square matrices M of order 9 with entries in $\{1, \ldots, 9\}$ that show valid sudoku grids.[1] How many of them have the property that the symmetric matrix $M + M^t$ is positive definite?

Proposed by Noam D. Elkies (Harvard University).

Solution by the proposer. We show that there are 0 such matrices. We prove this by showing that every such matrix satisfies $w(M + M^t)w^t = 0$ where w is the nonzero vector

$$w = (1, 1, 1, -1, -1, -1, 0, 0, 0).$$

Let $e_1, \ldots, e_9$ be the standard unit vectors in $\mathbb{R}^9$, and for each of $j = 1, 2, 3$ let $v_j = e_{3j-2} + e_{3j-1} + e_{3j}$, so

$$v_1 = (1, 1, 1, 0, 0, 0, 0, 0, 0),$$
$$v_2 = (0, 0, 0, 1, 1, 1, 0, 0, 0),$$
$$v_3 = (0, 0, 0, 0, 0, 0, 1, 1, 1).$$

Then for $j, k \in \{1, 2, 3\}$ we have

$$v_j M v_k^t = v_j M^t v_k^t = \sum_{i=1}^{9} i = 45,$$

because $v_j M v_k^t$ and $v_j M^t v_k^t$ are the sum of the entries in the (j, k)-th and (k, j)-th 3×3 block of the Sudoku array M. It follows that the vector $w = v_1 - v_2$ satisfies $wMw^t = wM^tw^t = 0$, whence $w(M + M^t)w^t = 0$ as claimed.

[1]The proposer points out that this calculation is detailed in Bertram Felgenhauer and Frazer Jarvis: Enumerating possible Sudoku grids (2005), `http://www.afjarvis.staff.shef.ac.uk/sudoku/sudoku.pdf`, athough it was independently computed by user "QSCGZ" on the rec.puzzle Google group, thread "combinatorial question on 9x9," 21 Sep. 2003.

Remark. We could have used for w any nonzero vector in the 2-dimensional space

$$V = \{a_1v_1 + a_2v_2 + a_3v_3 \mid a_1 + a_2 + a_3 = 0\}.$$

It follows that if $M + M^t$ is positive *semi*definite then its kernel contains V. We have not found any such M, but neither can we prove that none exists. □

Also solved by Daniel Kane, G2.

Diophantine Squeeze

S08 – 3. Let $k \geq 1$ be a natural number. Find all integer solutions to the diophantine equation

$$x^{2k+1} + x^{2k} + \cdots + x^2 + x + 1 = y^{2k+1}.$$

Proposed by Ovidiu Furdui (University of Toledo).

Solution by the Missouri State University Problem Solving Group. Clearly, $(x, y) = (0, 1)$ or $(-1, 0)$ are solutions for all k. We claim that there are no other solutions. Note that

$$\sum_{i=0}^{2k} x^i = \begin{cases} (x^{2k+1} - 1)/(x - 1) & \text{if } x \neq 1 \\ 2k + 1 & \text{if } x = 1 \end{cases}$$

which is positive for all x. Therefore

$$x^{2k+1} < \sum_{i=0}^{2k+1} x^i \text{ for all } x.$$

If $x > 0$, then

$$x^{2k+1} < \sum_{i=0}^{2k+1} x^i = y^{2k+1} < \sum_{i=0}^{2k+1} \binom{2k+1}{i} x^i = (x + 1)^{2k+1}$$

which is clearly impossible for integers x and y.

If $x < -1$, we will prove that $(-1)^n \sum_{i=0}^{n} x^i > (-1)^n(x + 1)^n$ for $n \geq 2$ by induction on n. Since $x < 0$, this is clearly true when $n = 2$. Assume that the result holds when $n = m$, *i.e.*

$$(-1)^m \frac{x^{m+1} - 1}{x - 1} = (-1)^m \sum_{i=0}^{m} x^i > (-1)^m(x + 1)^m.$$

Multiplying both sides by $-(x + 1)$ (which is positive), we obtain

$$(-1)^{m+1} \frac{x^{m+2} + x^{m+1} - x - 1}{x - 1} > (-1)^{m+1}(x + 1)^{m+1}.$$

Now since $x < -1$, $(-1)^{m+1}(x^{m+1} - x)/(x - 1)$ is negative regardless of whether m is even or odd, so

$$\begin{aligned}(-1)^{m+1} \sum_{i=0}^{m+1} x^i &= (-1)^{m+1} \frac{x^{m+2} - 1}{x - 1} \\ &> (-1)^{m+1} \frac{x^{m+2} + x^{m+1} - x - 1}{x - 1} \\ &> (-1)^{m+1}(x + 1)^{m+1}\end{aligned}$$

which is what we needed to show.

We are interested in the case $n = 2k + 1$ where the inequality we just proved becomes

$$\sum_{i=0}^{2k+1} x^i < (x+1)^{2k+1}.$$

As in the case when $x > 0$, we have

$$x^{2k+1} < \sum_{i=0}^{2k+1} x^i = y^{2k+1} < (x+1)^{2k+1}$$

which is again impossible. □

Also solved by the Northwestern University Problem Solving Group, Koichiro Nomura (University of Tokyo, Japan) and the proposer.

E. Equilateralibus Isosceles

S08 – 5. Let ABC be a non-isosceles triangle with $\angle A = 60°$. Let H be its orthocenter and I its incenter. Let B_i and C_i the points such that the equilateral triangles ABC_i and AB_iC intersect the interior of ABC. Define B_e and C_e similarly, so that ABC_e and AB_eC are equilateral and disjoint from the interior of ABC.

Show that the lines through HI, B_iC_i and B_eC_e do not concur, and that the triangle they form is isosceles.

Proposed by Daniel Campos Salas (Costa Rica).

Solution by Yasuhide Minoda (Tetsuryokukai Institute, Japan). Without loss of generality, we may assume $AB < AC$. Let $X = B_iC_i \cap AB_e$. Since $\angle CBC_i = \angle BC_iA - \angle BCC_i = \frac{\pi}{3} - \angle C$ and

$$\begin{aligned} \angle CBC_i &= \angle CB_iC_i \quad \text{(because } B_i, B, C_i, C \text{ are concyclic)} \\ &= \angle B_iXA \quad \text{(because } B_iC \text{ and } AX \text{ are parallel),} \end{aligned}$$

we have $\angle B_iXA = \frac{\pi}{3} - \angle C$.

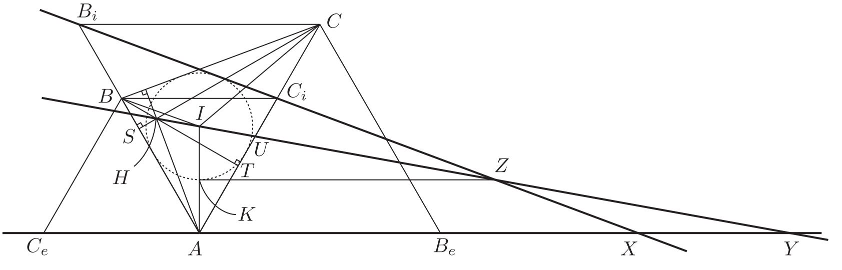

Figure 12.1: Diagram for Problem S08–5.

Next, let $Y = HI \cap AB_e$; we now show $\angle IYA = \frac{1}{2}\angle B_iXA$. Let $S = CH \cap AB$ and $T = BH \cap AC$. We have $\angle BHC = \angle SHT = 2\pi - \frac{\pi}{2} - \frac{\pi}{2} - \angle A = \frac{2\pi}{3}$ by considering quadrilateral $HSAT$, and from triangle IBC it follows that

$$\angle BIC = \pi - (\angle IBC + \angle ICB) = \pi - \frac{1}{2}(\angle B + \angle C) = \pi - \frac{1}{2}(\pi - \angle A) = \frac{2\pi}{3}.$$

It follows that $\angle BHC = \angle BIC$, from which we conclude (letting $U = HI \cap AC$):

$$\begin{aligned} & B, H, I, C \text{ are concyclic} \\ \implies & \angle IHT = \angle ICB = \frac{1}{2}\angle C \\ \implies & \angle CUY = \angle HUT = \frac{\pi}{2} - \angle IHT = \frac{\pi}{2} - \frac{1}{2}\angle C \\ \implies & \angle IYA = \angle CUY - \angle UAY = \frac{\pi}{2} - \frac{1}{2}\angle C - \frac{\pi}{6} = \frac{\pi}{6} - \frac{1}{2}\angle C \\ \implies & IYA = \frac{1}{2}\angle B_iXA. \end{aligned}$$

Therefore, if HI, B_iC_i, and B_eC_e do not concur, the triangle they form is isosceles.

Now we have to show that HI, B_iC_i and B_eC_e do not concur. Draw a tangent line to the incircle of triangle ABC from $Z = HI \cap B_iC_i$, other than ZB_i, and let K be the tangent point. Clearly $\angle B_iZI = \angle IZK$. Since $\angle B_iZI = \angle IYA$, we have $\angle IZK = \angle IYA$. Thus ZK and AY are parallel, so K is the intersection of the incircle of triangle ABC and the segment IA. It is clear that $K \neq A$, so we have $Z \neq X$. This means that HI, B_iC_i, and B_eC_e do not concur. □

Also solved by Koichiro Nomura (University of Tokyo, Japan) and the proposer.