FEATURE

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STATISTICS CORNER

Conformal Invariance in the Scaling Limit of Critical Planar Percolation

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8.1 Introduction to percolation theory

Percolation theory was originally developed to model the flow of liquid through a disordered porous medium. The classic example comes from coffee-making: If water is poured through coffee grounds, we would like to find out what the wet portion of the grounds might look like.

We can model the material as a graph Λ, with vertex set V and edge set E. (We will be concerned only with the case where Λ is undirected.) Each vertex is a particle of coffee, and edges join vertices corresponding to adjacent particles. In the standard terminology of percolation theory, vertices and edges are referred to as **sites** and **bonds** respectively. We can then model percolation as a random binary function on the graph: In **site percolation**, each site is independently set to be **open** or **closed** (wet or dry) with probability p; the open sites induce the **open subgraph** of Λ. **Bond percolation** is defined analogously, the only modification being that we select a random subset of **open bonds** of E. A choice of open and closed sites (or bonds) is called a **(percolation) configuration**. Percolation theory is specifically concerned with the connected components of the open subgraph, or **open clusters** — the wet portions of the coffee which are now sticking together.

For further background on this subject see Grimmett [Gr] and Bollobás and Riordan [BR]. Besides giving us insights into what happens inside our coffee-makers, percolation theory has been applied to study earthquakes and fault patterns, groundwater flow in rock, reactions in evolving porous media, random electrical networks, and semiconductors [Sa].

The goal of this article is to describe a surprising connection between the scaling limit of percolation and conformal maps. This article assumes some knowledge of complex analysis (as in [Ah] or [SS]) and basic probability theory.

8.1.1 Critical percolation probabilities

Let $\mathbb{P}_p$ denote the probability measure induced by percolation at probability p on the space of subgraphs of Λ. (In simpler terms, for any event E which is determined by the states of any or all of the sites in the graph — for example, the event that there is an infinite open cluster — $\mathbb{P}_p(E)$ denotes the probability that E will occur if each site is chosen to be open at probability p.) For $x \in \Lambda$ we can consider C_x, the **open cluster at** x, which is the connected component of the open subgraph containing x. (In particular, $C_x = \varnothing$ if and only if x is closed.) We define two quantities of interest at x:

$$\theta_x(p) = \mathbb{P}_p(|C_x| = \infty) \text{ and } \chi_x(p) = \mathbb{E}_p(|C_x|),$$

where $\mathbb{E}_p$ denotes expectation with respect to $\mathbb{P}_p$. θ_x and χ_x are nondecreasing in p. If Λ is a connected graph, then for any $x, y \in \Lambda$, $\theta_x(p)$ and $\theta_y(p)$ must be both positive or both zero,

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and $\chi_x(p)$ and $\chi_y(p)$ must be both finite or both infinite. Therefore we can define two **critical probabilities** for the graph Λ,

$$p_H = \inf\{p : \theta_x(p) > 0\}, \;\; p_T = \inf\{p : \chi_x(p) = \infty\}.$$

(The subscript of p_H refers to Hammersley while that of p_T refers to Temperley.) If $p > p_H$, at every $x \in \Lambda$ there is a positive probability that C_x is infinite; in particular, by the Kolmogorov 0-1 Law, an infinite connected component exists with probability 1 (see e.g. [Ro]).

We always have $p_T \leq p_H$ for a given percolation model; $p_T < p_H$ can occur if, for some values of p, $|C_x|$ is finite with probability 1 but has a heavy-tailed distribution. Menshikov proves that under some uniformity conditions on the structure of Λ, for $p < p_H$ the distribution of $|C_x|$ has an almost exponential tail, so that $\chi_x(p)$ must be finite; and this is enough to conclude $p_T = p_H$. For example, $p_T = p_H = 1/2$ for Λ equal to $\mathbb{Z}^2$ or T (the triangular lattice) [BR]. We will see later that the most interesting behavior occurs for models at critical probabilities — when the medium is neither under- nor over-saturated.

8.1.2 Crossing probabilities in the scaling limit

For the remainder of this discussion, we will restrict ourselves to **site percolation on a planar lattice**. Planar lattices all satisfy the conditions of Menshikov's theorem, so their two critical probabilities are equal, and we denote them both by p_c. In particular, the main result of this section, due to Smirnov, concerns the triangular lattice T, shown with its dual hexagonal lattice H in Figure 8.1. Site percolation on a lattice can be visualized as **face percolation** (the shaded hexagons) on its dual.

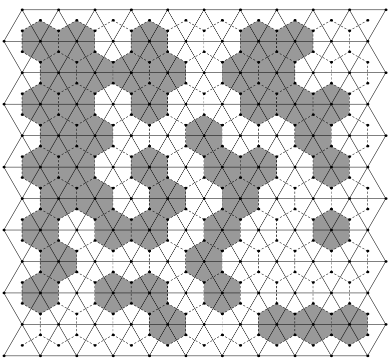

Figure 8.1: Triangular lattice T, hexagonal dual H

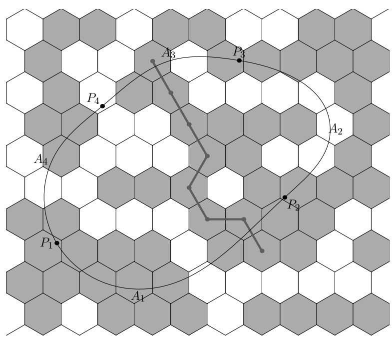

Figure 8.2: Open crossing of D (dark) in lattice δT

Let $D \subset \mathbb{C}$ be a simply connected, bounded domain, whose boundary is a Jordan curve Γ. Let z_i, $1 \leq i \leq 4$, be distinct boundary points of D, appearing in this cyclic order as Γ is traversed counterclockwise; then $D_4 = (D; z_1, z_2, z_3, z_4)$ is a **4-marked domain**. Let $A_i = A_i(D_4)$ be the arc of Γ from z_i to z_{i+1}, where the indices are taken modulo 4.

Let Λ be a planar lattice; we wish to analyze the structure of percolation on $\delta\Lambda$, the rescaled lattice, within D_4 as $\delta \to 0$. This is the notion of a **scaling limit**. By an **open crossing of D from A_1 to A_3 in $\delta\Lambda$**, we mean an open path $v_0 v_1 \cdots v_t$ in $\delta\Lambda$ such that $v_1, \cdots, v_{t-1} \in D$, $v_0, v_t \notin D$, and $v_0 v_1$ meets A_1 while $v_{t-1} v_t$ meets A_3; see Figure 8.2. Considering either site or bond percolation on Λ, for $\delta > 0$ and $0 < p < 1$, we can then define

$$P_\delta(D_4, \Lambda, p) = \mathbb{P}_p(D_4 \text{ has an open crossing in } \delta\Lambda).$$

We are then interested in $\lim_{\delta \to 0} P_\delta(D_4, \Lambda, p)$.

In fact, for almost all values of p the limit is uninteresting: Menshikov's theorem can be applied to show that for $p < p_c$ the limit is 0, and for $p > p_c$ the limit is 1. The only interesting behavior occurs when $p = p_c$; we therefore restrict ourselves to **critical percolation**, and denote $\mathbb{P}_\delta(D_4, \Lambda) = \mathbb{P}_\delta(D_4, \Lambda, p_c)$, and $\pi(D_4, \Lambda) = \lim_{\delta\to 0} P_\delta(D_4, \Lambda)$. Based on experiments, Langlands, Pouliot, and Saint-Aubin [LPSA] made the following conjecture:

Conjecture 1 ([LPSA])**.** *The limit $\pi(D_4, \Lambda)$ is defined, lies in $(0, 1)$, and is **conformally invariant**: If $D_4 = (D; z_1, z_2, z_3, z_4)$ and $D'_4 = (D', z'_1, z'_2, z'_3, z'_4)$ are conformally equivalent 4-marked domains (there is a conformal map $\varphi : D \to D'$ which extends continuously to ∂D with $P_i \mapsto P'_i$), then $\pi(D_4, \Lambda) = \pi(D'_4, \Lambda)$.*

To date this result has only been proven for the triangular lattice T, a fairly recent result due to Smirnov [Sm, Sm]. We will discuss this result, and a more general result also due to Smirnov, below. However, we first provide some motivation for why the scaling limit should be conformally invariant.

8.2 Conformal invariance of planar Brownian motion

Brownian motion is the scaling limit of **simple random walk** on $\mathbb{Z}$, the process which starts at some integer and at each step moves by ± 1 with equal probability, independently of all previous steps. Brownian motion is a **continuous stochastic process** $(B_t)_{t\geq 0}$ with normally distributed increments (e.g. by the central limit theorem), and with disjoint increments independent. The standard definition takes $B_t \sim \mathcal{N}(0, t)$, so that $B_t - B_s \sim \mathcal{N}(0, t - s)$ for $t > s$. A very readable introduction to Brownian motion can be found in Steele's book [St]; see also [RW, Va].

Now let $(B_t)_{t\geq 0}$ denote **complex Brownian motion** started at some $z \in \mathbb{C}$, that is, the real and imaginary parts B^1_t, B^2_t of (B_t) are independent Brownian motions. We will denote the probability measure for (B_t) by $\mathbb{P}_z$, where the subscript denotes the starting point of the process.

A translation of $\mathbb{C}$ of course maps (B_t) to another complex Brownian motion. Also, since the bivariate normal distribution has rotational symmetry, a rotation of the complex plane maps (B_t) to another complex Brownian motion. In fact, we can say much more. The following observation, due to Paul Lévy, essentially tells us that complex Brownian motion behaves as nicely as possible with respect to holomorphic transformations of their domain.

Theorem 2 ([La, Le])**.** *Let U be a domain in $\mathbb{C}$ with $z_0 \in U$, and let B_t be a complex Brownian motion started at z_0. Set $\tau_U = \inf\{t \geq 0 \mid B_t \notin U\}$ to be the hitting time of $\mathbb{C}\setminus U$. Let $f : U \to \mathbb{C}$ be a non-constant holomorphic map, and define $Y_t = f(B_t)$ for $0 \leq t \leq \tau_U$. Then*

$$Y_{\sigma(t)}, \text{ where } \sigma^{-1}(t) = \int_0^t |f'(B_s)|^2\, ds,$$

has the distribution of a standard Brownian motion.

Proof. We sketch the basic argument given by Lévy [Le], using the heuristic that a stochastic process is determined by its behavior locally near every point. The result is then intuitive since locally near B_t, the map f resembles rotation-dilation by $\lambda_t = f'(B_t)$. Rotations of Brownian motions are still Brownian motions, so locally Y_t looks like $|\lambda_t| B_t$. For $c \in \mathbb{R}$, if B_t is complex Brownian motion, then $cB_{t/c^2} = B''_t$ is another complex Brownian motion, by a simple calculation. Since $Y_{\sigma(t)}$ locally looks like $|\lambda_{\sigma(t)}| B_{\sigma(t)}$, $\sigma(t)$ should locally look like $t/|\lambda_{\sigma(t)}|^2$, that is, we should set $\sigma'(t) = 1/|f'(B_{\sigma(t)})|^2$. This is accomplished if $(\sigma^{-1})'(t) = |f'(B_t)|^2$. □

In words, the theorem says that **a holomorphic map of a complex Brownian motion is another complex Brownian motion**, up to a (random) time change. In particular, the curve traced out by $f(B_t)$, $0 \leq t \leq \tau_U$, is indistinguishable from a curve traced out by a complex Brownian motion. Most modern proofs of this result use Itō's lemma, together with harmonicity of the real and imaginary parts of φ [Ga, La]. Because Brownian motion occurs as the scaling limit of simple random walk, this result is one of the strongest motivations for the study of conformally invariant scaling limits coming from general discrete processes.

A further connection between Brownian motion and harmonic functions lies in the following result, due to Kakutani [Ka]. The form of the statement below is from Lawler [La], and shows that Brownian motion gives a solution to the **Dirichlet problem**.

Proposition 3 ([Ka, La])**.** *Let $D \subset \mathbb{C}$ be a bounded Jordan domain (not necessarily simply connected), and let $f : \partial D \to \mathbb{R}$ be bounded and measurable. Let $\tau_D = \inf\{t > 0 : B_t \notin D\}$ be the hitting time of $\mathbb{C} \setminus D$. Define $u : \overline{D} \to \mathbb{R}$ by*

$$u(z) = \begin{cases} f(z) & \text{if } z \in \partial D. \\ \mathbb{E}_z f(B_{\tau_D}) & \text{if } z \in D. \end{cases}$$

Then u is a bounded, harmonic function in D and is continuous at all points $z \in \partial D$ at which f is continuous.

Proof. $\|u\|_\infty \leq \|f\|_\infty < \infty$ (by assumption), so u is bounded. To show that u is harmonic, it suffices to check the mean-value property (see [Ah]),

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta,$$

for any $z \in D$ and any $r > 0$ such that $\overline{D_r(z)} = \{|w - z| \leq r\}$ is contained in D. Let σ denote the hitting time of $\mathbb{C} \setminus D_r(z_0)$. By the rotational symmetry of Brownian motion, B_σ has the uniform distribution on $\partial D_r(z)$ with respect to $\mathbb{P}_z$, so the above equals

$$\mathbb{E}_z(u(B_\sigma)) = \mathbb{E}_z\left[\mathbb{E}_{B_\sigma}(f(B_{\tau_D}))\right] = \mathbb{E}_z\left[\mathbb{E}_z(f(B_{\tau_D})|B_\sigma)\right] = \mathbb{E}_z f(B_{\tau_D}) = u(z),$$

where the second equality uses the strong Markov property, and the third is the law of iterated expectations. This proves that u is harmonic, and the continuity result follows from the boundedness of f together with the regularity of the domain. □

Brownian motion is the simplest non-trivial example of the scaling limit of a discrete process, and yet most known examples of stochastic processes are related to Brownian motion. Kakutani's result therefore suggests a very general connection between stochastic processes and complex analysis.

8.3 Smirnov's theorem

Smirnov [Sm, Sm] proves conformal invariance of crossing probabilities for the triangular lattice. The proof approximates D by a succession of 4-marked **discrete domains** G_δ in δT: Each $G = G_\delta$ has vertices v_i ($1 \leq i \leq 4$) marked on its internal boundary to approximate the P_i; the vertices demarcate arcs $A_i(G)$ on the internal boundary, and the external boundary can be partitioned into corresponding arcs $A_i^+(G)$.

The main idea is to express crossing probabilities for a 4-marked discrete domain G_4 in terms of **separating probabilities** for the 3-marked discrete domain $G_3 = (G; v_1, v_2, v_3)$ obtained by dropping v_4: For $z \in \delta H \cap D$ (so $z \in D$ is the center of a triangle in δT), we can define

$$s_\delta^i(z) = \mathbb{P}(\exists \text{ open } A_{i-1}(G_3)\text{–}A_i(G_3) \text{ path in } \delta\Lambda \text{ separating } z \text{ from } A_{i+1}^+(G_3)),$$

for $i = 1, 2, 3$, where the indices are taken modulo 3. That is, $s_\delta^i(z)$ is the probability there is a (simple) open path joining the two boundary arcs meeting v_i, separating z from the external boundary arc opposite v_i. In particular, **for z lying near v_4, $s_\delta^1(z)$ is almost exactly the crossing probability for the original 4-marked domain.**

The discrete derivatives of the s_δ^i satisfy a $2\pi/3$-rotational version of the Cauchy-Riemann equations. As a result it can be shown that the s_δ^i (extended by interpolation to continuous functions on $\overline{D}$) converge uniformly to a **"harmonic conjugate triple"** (s^1, s^2, s^3) satisfying a mixed Dirichlet problem on D, which has a unique and conformally invariant solution.

An extremely detailed proof of Smirnov's theorem, with full justification for the approximation by discrete domains, is presented in [BR].

8.4 Conclusion

In fact, crossing probabilities are only the simplest possible example of a conformal invariant. Smirnov also proves the much more general result that the **“full percolation configuration”** is conformally invariant in the scaling limit: If we encode a particular configuration by a collection of curves (for example, by the external perimeters of connected components), then the probability laws of these curves are conformally invariant in the scaling limit of the lattice. This shows for instance that the **percolation interface**, depicted in Figures 8.3 and 8.4, follows a conformally invariant probability law in the scaling limit just as Brownian motion does.

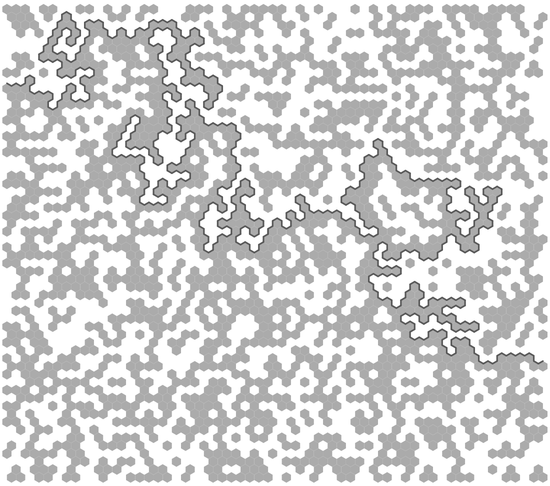

Figure 8.3: A portion of the percolation interface

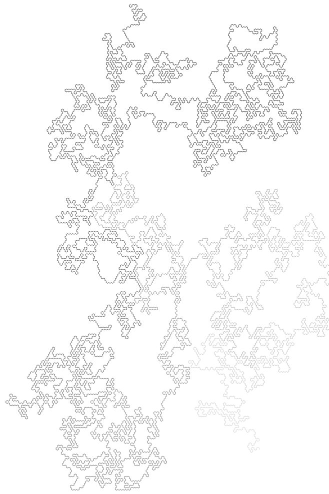

Figure 8.4: At a smaller mesh

Proving this result requires more technical work on the convergence of probability laws for random curves, so we refer the interested reader to the paper by Aizenman and Burchard [AB], and to Smirnov's original paper. We note only that this result differs significantly from Lévy's result for Brownian motion because the curve is **self-avoiding**,[1] and therefore does not have independent increments, so the result has surprisingly strong implications. For example, if γ is a conformally invariant self-avoiding **stochastic process** which starts on the boundary of a simply connected domain D and travels in D, by the Riemann mapping theorem there is a map φ which takes $D \setminus \gamma[0,t]$ conformally onto D, with γ_t mapped by the continuous extension to γ_0. The curve $\varphi(\gamma[t,\infty))$ then follows the *same* law that the original curve γ followed. This observation is a starting point for the theory of **Schramm-Loewner evolutions**; for further reading see [Sc, We].

8.5 Acknowledgements

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